Hypergeometrics and $1/\pi^2$

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Notation

In what follows, I use the standard notation for the hypergeometric series,

$$ {}_mF_{m-1}\left(\begin{matrix} a_1,\ a_2,\ \dots,\ a_m \\ b_2,\ \dots,\ b_m \end{matrix} \,\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_m)_n}{(b_2)_n \cdots (b_m)_n} \frac{z^n}{n!}, $$

where

$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \prod_{j=1}^{n-1} (a+j) $$

denotes the Pochhammer symbol (or rising factorial).

Special sequence

The sequence

$$a_n = \sum_{k=0}^{n} \frac{(\frac{1}{2})_k^3}{k!^3} \frac{(\frac{1}{2})_{n-k}}{(n-k)!} = \sum_{k=0}^{n} \left(\frac{(\frac{1}{4})_k (\frac{3}{4})_{n-k}}{k!(n-k)!} \right)^2$$

$$= 2^{-6n} \sum_{k=0}^{n} \binom{2k}{k}^3 \binom{2n-2k}{n-k} 2^{4(n-k)}$$

has a generating function

$$g(z) = {}_2F_1\left(\begin{matrix} \frac{1}{4}, \frac{1}{4} \\ 1 \end{matrix} \,\middle|\, z \right) \cdot {}_2F_1\left(\begin{matrix} \frac{3}{4}, \frac{3}{4} \\ 1 \end{matrix} \,\middle|\, z \right)$$

$$= \frac{1}{1-z} \, {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{4}, \frac{3}{4} \\ 1, 1 \end{matrix} \,\middle|\, \frac{-4z}{(1-z)^2} \right),$$

satisfies $A_n = 2^{6n} a_n \in \mathbb{Z}$ (trivially) and

$$(n+1)^3 A_{n+1} - 8(2n+1)(8n^2+8n+5) A_n + 4096 n^3 A_{n-1} = 0, \qquad n = 1, 2, \dots .$$

Numerical identities

Here are examples of series for $1/\pi^2$:

$$\sum_{n=0}^{\infty} a_n \frac{(\frac{1}{2})_n^2}{n!^2}(18n^2+7n+1)\left(\frac{-1}{2}\right)^{3n} \stackrel{?}{=} \frac{4\sqrt{2}}{\pi^2},$$

$$\sum_{n=0}^{\infty} a_n \frac{(\frac{1}{3})_n(\frac{2}{3})_n}{n!^2}(198144387n^2+28855107n+1400726)\left(\frac{-1}{80}\right)^{3n} = \frac{240^3}{\pi^2},$$

$$\sum_{n=0}^{\infty} a_n \frac{(\frac{1}{4})_n(\frac{3}{4})_n}{n!^2}(18n^2-10n-3)\left(\frac{16}{25}\right)^{n} = \frac{10\sqrt{5}}{\pi^2}.$$

The first one was found experimentally by Zhi-Wei Sun (in 2011), while the latter two were proven by myself 4–5 years ago.

Problem 1

My proofs of the two formulas are based on the hypergeometric identities

$$ {}_3F_2\left(\begin{matrix}\frac12,\ \frac16,\ \frac56\\ 1,\ 1\end{matrix}\ \middle|\ z\right)^2=\sum_{n=0}^\infty a_n\frac{(\frac13)_n(\frac23)_n}{n!^2}z^n $$

and

$$ {}_5F_4\left(\begin{matrix}\frac12,\ \frac12,\ \frac12,\ \frac12,\ \frac12\\ 1,\ 1,\ 1,\ 1\end{matrix}\ \middle|\ z\right)=\frac{1}{(1-z)^{1/2}}\sum_{n=0}^\infty a_n\frac{(\frac14)_n(\frac34)_n}{n!^2}\left(\frac{-4z}{(1-z)^2}\right)^n. $$

Problem 1

Is there a purely hypergeometric expression for

$$ \sum_{n=0}^\infty a_n\frac{(\frac12)_n^2}{n!^2}z^n $$

(up to algebraic transformation of the argument)?

Problem 2

The natural quest for the first hypergeometric identity

$$ {}_3F_2\left(\begin{matrix} \frac12, \frac16, \frac56 \\ 1, 1 \end{matrix} \,\middle|\, z\right)^2 = \sum_{n=0}^{\infty} a_n \frac{(\frac13)_n(\frac23)_n}{n!^2} z^n, $$

which has only single parameter z involved, is the following.

Problem 2

Give a version of the identity which depends on more than one parameter. For example, extend it to

$$ \sum_{n=0}^{\infty} a_n \frac{(s)_n(1-s)_n}{n!^2} z^n, $$

which is related to the previous problem, or to the product

$$ {}_3F_2\left(\begin{matrix} \frac12, \frac16, \frac56 \\ 1, 1 \end{matrix} \,\middle|\, z\right) \cdot {}_3F_2\left(\begin{matrix} \frac12, \frac16, \frac56 \\ 1, 1 \end{matrix} \,\middle|\, w\right). $$