Generating functions of Legendre polynomials

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Overview

Legendre polynomials

Consider the Legendre polynomials $P_n(x)$,

$$P_n(x) = {}_2F_1\left(\begin{matrix} -n,\ n+1 \\ 1 \end{matrix} \,\middle|\, \frac{1-x}{2}\right) = \left(\frac{x+1}{2}\right)^n {}_2F_1\left(\begin{matrix} -n,\ -n \\ 1 \end{matrix} \,\middle|\, \frac{x-1}{x+1}\right)$$

$$= \sum_{m=0}^{n} \binom{n}{m}^2 \left(\frac{x-1}{2}\right)^m \left(\frac{x+1}{2}\right)^{n-m},$$

where I use a standard notation for the hypergeometric series,

$${}_mF_{m-1}\left(\begin{matrix} a_1,\ a_2,\ \dots,\ a_m \\ b_2,\ \dots,\ b_m \end{matrix} \,\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_m)_n}{(b_2)_n \cdots (b_m)_n} \frac{z^n}{n!},$$

and $(a)_n = \Gamma(a+n)/\Gamma(a)$ denotes the Pochhammer symbol (or rising factorial).

Brafman's generating function

The Legendre polynomials can be alternatively given by the generating function

$$(1-2xz+z^2)^{-1/2}=\sum_{n=0}^{\infty}P_n(x)z^n,$$

but there are many other generating functions for them. One particular family of examples is due to F. Brafman (1951).

Theorem A

The following generating series is valid:

$$\sum_{n=0}^{\infty}\frac{(s)_n(1-s)_n}{n!^2}P_n(x)z^n={}_2F_1\biggl(\begin{matrix} s,\ 1-s \\ 1 \end{matrix}\biggm| \frac{1-\rho-z}{2}\biggr)\cdot{}_2F_1\biggl(\begin{matrix} s,\ 1-s \\ 1 \end{matrix}\biggm| \frac{1-\rho+z}{2}\biggr),$$

where $\rho=\rho(x,z):=(1-2xz+z^2)^{1/2}$.

Bailey's identity

Theorem A in the form

$$\sum_{n=0}^{\infty}\frac{(s)_n(1-s)_n}{n!^2}P_n\biggl(\frac{X+Y-2XY}{Y-X}\biggr)(Y-X)^n = {}_2F_1\biggl(\begin{matrix} s,\,1-s \\ 1\end{matrix}\biggm| X\biggr)\cdot{}_2F_1\biggl(\begin{matrix} s,\,1-s \\ 1\end{matrix}\biggm| Y\biggr)$$

is derived by Brafman as a consequence of Bailey's identity for a special case of Appell's hypergeometric function of the fourth type,

$$\sum_{m,k=0}^{\infty}\frac{(s)_{m+k}(1-s)_{m+k}}{m!^2k!^2}\bigl(X(1-Y)\bigr)^m\bigl(Y(1-X)\bigr)^k = {}_2F_1\biggl(\begin{matrix} s,\,1-s \\ 1\end{matrix}\biggm| X\biggr)\cdot{}_2F_1\biggl(\begin{matrix} s,\,1-s \\ 1\end{matrix}\biggm| Y\biggr).$$

Clausen's identity

Note that by specializing $Y = X$, one recovers a particular case of Clausen's identity:

$$ {}_3F_2\left(\begin{matrix} \frac{1}{2}, \, s, \, 1-s \\ 1, \, 1 \end{matrix} \,\middle|\, 4X(1-X)\right) = {}_2F_1\left(\begin{matrix} s, \, 1-s \\ 1 \end{matrix} \,\middle|\, X\right)^2. $$

Brafman–Srivastava theorem

Theorem B (Brafman (1959), Srivastava (1975))

For a positive integer N, a (generic) sequence $\lambda_0, \lambda_1, \dots$ and a complex number w,

$$\frac{1}{\rho}\sum_{k=0}^{\infty}\lambda_k P_{Nk}\left(\frac{x-z}{\rho}\right)\left(w\frac{z^N}{\rho^N}\right)^k = \sum_{n=0}^{\infty} A_n P_n(x) z^n,$$

where $\rho = (1-2xz+z^2)^{1/2}$ and

$$A_n = A_n(w) = \sum_{k=0}^{\lfloor n/N \rfloor} \binom{n}{Nk} \lambda_k w^k.$$

Brafman's original results address the cases $N = 1, 2$ and a sequence λ_n given as a quotient of Pochhammer symbols (in modern terminology, λ_n is called a *hypergeometric term*).

Apéry-like sequences

We extend Bailey's identity and Brafman's generating function to more general Apéry-like sequences $u_0, u_1, u_2, \dots$ which satisfy the second order recurrence relation

$$(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1} \quad \text{for } n = 0, 1, 2, \dots, \\ u_{-1} = 0, \quad u_0 = 1,$$

for a given data a, b and c.
The hypergeometric term $u_n = (s)_n(1-s)_n/n!^2$ corresponds to a special degenerate case $c = 0$ and $a = 1$, $b = s(1-s)$ in the recursion.
Note that the generating series $F(X) = \sum_{n=0}^{\infty} u_n X^n$ for a sequence satisfying the recurrence equation is a unique, analytic at the origin solution of the differential equation

$$\bigl(\theta^2 - X(a\theta^2 + a\theta + b) + cX^2(\theta+1)^2\bigr)F(X) = 0, \qquad \text{where} \quad \theta = \theta_X := X\frac{\partial}{\partial X}.$$

Gist 1: Generalized Bailey's identity

Our first result concerns the generating function of u_n.

Theorem 1

For the solution u_n of the recurrence equation above, define

$$g(X,Y)=\frac{X(1-aY+cY^2)}{(1-cXY)^2}.$$

Then in a neighbourhood of $X=Y=0$,

$$\biggl\{\sum_{n=0}^\infty u_nX^n\biggr\}\biggl\{\sum_{n=0}^\infty u_nY^n\biggr\}=\frac{1}{1-cXY}\sum_{n=0}^\infty u_n\sum_{m=0}^n\binom{n}{m}^2g(X,Y)^mg(Y,X)^{n-m}.$$

Therefore, Bailey's identity corresponds to the particular choice $c=0$ in Theorem 1.

Gist 2: Generalized Brafman's identity

Theorem 1 also generalizes Clausen-type formulae given recently by H. H. Chan, Y. Tanigawa, Y. Yang, and W. Z.; they arise as specialization $Y = X$.

Following Brafman's derivation of Theorem A we deduce the following generalized generating functions of Legendre polynomials.

Theorem 2

For the solution u_n of the recurrence equation above, the following identity is valid in a neighbourhood of $X = Y = 0$:

$$\sum_{n=0}^{\infty} u_n P_n\biggl(\frac{(X+Y)(1+cXY)-2aXY}{(Y-X)(1-cXY)}\biggr)\biggl(\frac{Y-X}{1-cXY}\biggr)^n = (1-cXY)\biggl\{\sum_{n=0}^{\infty} u_n X^n\biggr\}\biggl\{\sum_{n=0}^{\infty} u_n Y^n\biggr\}.$$

Gist 3: Special generating functions

Theorem 3

The following identities are valid in a neighbourhood of $X = Y = 0$:

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{n!^2} P_{2n}\biggl(\frac{(1-X-Y)(X+Y-2XY)}{(Y-X)(1-X-Y+2XY)}\biggr) \cdot \biggl(\frac{X-Y}{1-X-Y+2XY}\biggr)^{2n}$$

$$= (1-X-Y+2XY)\,{}_2F_1\biggl(\begin{matrix}\frac{1}{2},\ \frac{1}{2}\\ 1\end{matrix}\biggm| 4X(1-X)\biggr) \cdot {}_2F_1\biggl(\begin{matrix}\frac{1}{2},\ \frac{1}{2}\\ 1\end{matrix}\biggm| 4Y(1-Y)\biggr),$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{3})_n(\frac{2}{3})_n}{n!^2} P_{3n}\biggl(\frac{(X+Y)(1-X-Y+3XY)-2XY}{(Y-X)\sqrt{p(X,Y)}}\biggr) \cdot \biggl(\frac{X-Y}{\sqrt{p(X,Y)}}\biggr)^{3n}$$

$$= \frac{\sqrt{p(X,Y)}}{(1-3X)(1-3Y)}\,{}_2F_1\biggl(\begin{matrix}\frac{1}{3},\ \frac{2}{3}\\ 1\end{matrix}\biggm| -\frac{9X(1-3X+3X^2)}{(1-3X)^3}\biggr)$$

$$\times\,{}_2F_1\biggl(\begin{matrix}\frac{1}{3},\ \frac{2}{3}\\ 1\end{matrix}\biggm| -\frac{9Y(1-3Y+3Y^2)}{(1-3Y)^3}\biggr),$$

where $p(X,Y) = (1-X-Y+3XY)^2 - 4XY$.

Fred Brafman

Fred Brafman was born on July 10, 1923 in Cincinnati, Ohio. He attended Lebanon High School (Ohio) from 1936 to 1940, then spent a year at Greenbrier Military School (Jr. College) before enrolling in the Engineering School at the University of Michigan in September 1941. He received a Bachelor of Science in Engineering (in Electrical Engineering) degree in 1943 and then a Bachelor of Science in Mathematics degree from Michigan in 1946.

Brafman entered the graduate program in Mathematics in the fall of 1946 and compiled an outstanding academic record. He received an AM degree in 1947 and a PhD in February 1951 from the University of Michigan under the supervision of E. D. Rainville. After completion of his PhD, he was hired by the Wayne State University, by the Southern Illinois University, and then by the University of Oklahoma. Brafman had an invitation to visit the Institute for Advanced Studies (Princeton) which was not materialized because of his ultimate death on February 4, 1959 in Oklahoma. He solely authored ten mathematical papers, all about special (orthogonal) polynomials.

Ramanujan's series for $1/\pi$

In 1914 S. Ramanujan recorded a list of 17 series for $1/\pi$, in particular,

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n (\frac{1}{2})_n (\frac{3}{4})_n}{n!^3} (21460n + 1123) \cdot \frac{(-1)^n}{882^{2n+1}} = \frac{4}{\pi},$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n (\frac{1}{2})_n (\frac{3}{4})_n}{n!^3} (26390n + 1103) \cdot \frac{1}{99^{4n+2}} = \frac{1}{2\pi\sqrt{2}}$$

which produce rapidly converging (rational) approximations to π.

Generalizations

An example is the Chudnovskys' famous formula which enabled them to hold the record for the calculation of π in 1989–94:

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{6})_n(\frac{1}{2})_n(\frac{5}{6})_n}{n!^3}(545140134n + 13591409) \cdot \frac{(-1)^n}{53360^{3n+2}} = \frac{3}{2\pi\sqrt{10005}}\,.$$

A more sophisticated example (which also shows that modularity rather than hypergeometrics is crucial) is T. Sato's formula (2002)

$$\sum_{n=0}^{\infty} u_n \cdot (20n + 10 - 3\sqrt{5})\left(\frac{\sqrt{5}-1}{2}\right)^{12n} = \frac{20\sqrt{3} + 9\sqrt{15}}{6\pi}$$

of Ramanujan type, involving Apéry's numbers

$$u_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \in \mathbb{Z}, \qquad n = 0, 1, 2, \dots,$$

which satisfy the recursion

$$(n+1)^3 u_{n+1} - (2n+1)(17n^2 + 17n + 5)u_n + n^3 u_{n-1} = 0.$$

News from 2011

Recently, Z.-W. Sun (and G. Almkvist) experimentally observed several new identities for $1/\pi$ of the form

$$\sum_{n=0}^{\infty} \frac{(s)_n(1-s)_n}{n!^2}(A+Bn)T_n(b,c)\lambda^n = \frac{C}{\pi},$$

where $s \in \{1/2, 1/3, 1/4\}$, $A, B, b, c \in \mathbb{Z}$, $T_n(b,c)$ denotes the coefficient of x^n in the expansion of $(x^2+bx+c)^n$, viz.

$$T_n(b,c) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k} b^{n-2k} c^k,$$

while λ and C are either rational or (linear combinations of) quadratic irrationalities.

Sun(ny) identities

Examples:

$$\sum_{n=0}^{\infty}\binom{2n}{n}^2(7+30n)\frac{T_n(34,1)}{(-2^{10})^n}=\frac{12}{\pi},$$

$$\sum_{n=0}^{\infty}\frac{(3n)!}{n!^3}(1+18n)\frac{T_n(730,729)}{30^{3n}}=\frac{25\sqrt{3}}{\pi},$$

$$\sum_{n=0}^{\infty}\frac{(4n)!}{(2n)!n!^2}(13+80n)\frac{T_n(7,4096)}{(-168^2)^n}=\frac{14\sqrt{210}+21\sqrt{42}}{8\pi},$$

$$\sum_{n=0}^{\infty}\binom{2n}{n}^2(1+10n)\frac{T_{2n}(38,1)}{240^{2n}}=\frac{15\sqrt{6}}{4\pi},$$

$$\sum_{n=0}^{\infty}\frac{(3n)!}{n!^3}(277+1638n)\frac{T_{3n}(62,1)}{(-240)^{3n}}=\frac{44\sqrt{105}}{\pi}.$$

History (following (the) Sun)

In the evening of Jan. 1, 2011 I figured out the asymptotic behavior of $T_n(b,c)$ with b and c positive. (Few days later I learned the Laplace-Heine asymptotic formula for Legendre polynomials and hence knew that my conjectural main term of $T_n(b,c)$ as $n \to +\infty$ is indeed correct.)

The story of new series for $1/\pi$ began with (I1) which was found in the early morning of Jan. 2, 2011 immediately after I waked up on the bed. On Jan 4 I announced this via a message to `Number Theory Mailing List` as well as the initial version of [S11a] posted to `arXiv`. In the subsequent two weeks I communicated with some experts on π-series and wanted to know whether they could prove my conjectural (I1). On Jan. 20, it seemed clear that series like (I1) could not be easily proved by the current known methods used to establish Ramanujan-type series for $1/\pi$.

Then, I discovered (II1) on Jan. 21 and (III3) on Jan. 29. On Feb. 2 I found (IV1) and (IV4). Then, I discovered (IV2) on Feb. 5. When I waked up in the early morning of Feb. 6, I suddenly realized a (conjectural) criterion for the existence of series for $1/\pi$ of type IV. Based on this criterion, I found (IV3), (IV5)-(IV10) and (IV12) on Feb. 6, (IV11) on Feb. 7, (IV13) on Feb. 8, (IV14)-(IV16) on Feb. 9, and (IV17) on Feb. 10. On Feb. 14 I discovered (I2)-(I4) and (III4). I found the sophisticated

Sun's trinomials

The binomial sums $T_n(b,c)$ can be expressed via the classical Legendre polynomials

$$P_n(x) = {}_2F_1\left(\begin{matrix} -n,\ n+1 \\ 1 \end{matrix} \,\middle|\, \frac{1-x}{2}\right)$$

by means of the formula

$$T_n(b,c) = (b^2-4c)^{n/2} P_n\left(\frac{b}{(b^2-4c)^{1/2}}\right),$$

so that the above equalities assume the form

$$\sum_{n=0}^{\infty} \frac{(s)_n(1-s)_n}{n!^2}(A+Bn)P_n(x_0)z_0^n = \frac{C}{\pi}.$$

Modular parametrization

I briefly indicate how the results above allow one to prove identities for $1/\pi$.
Suppose that we have an *arithmetic* sequence u_n satisfying the recurrence given earlier, and denote by

$$F(t) := \sum_{n=0}^{\infty} u_n t^n \quad \text{and} \quad G(t) := \sum_{n=0}^{\infty} u_n n t^n = t\,\frac{\mathrm{d}F}{\mathrm{d}t}$$

the corresponding generating function and its derivative.
Then there exists a *modular* function $t(\tau)$ on a congruence subgroup of $SL_2(\mathbb{Z})$ such that $F(t(\tau))$ is a weight 1 modular form on the subgroup.
In particular, for a quadratic irrationality τ_0 with $\operatorname{Im}\tau_0 > 0$, the value $t(\tau_0)$ is an algebraic number and, under some technical conditions on $|t(\tau_0)|$, there is a *Ramanujan-type series* of the form

$$aF^2(t(\tau_0)) + 2bF(t(\tau_0))G(t(\tau_0)) = \frac{c}{\pi},$$

where a, b and c are certain (effectively computable) algebraic numbers.

Calculus

Suppose furthermore that we have a functional identity of the form

$$\sum_{n=0}^{\infty} u_n P_{\ell n}(x) z^n = \gamma F(\alpha) F(\beta),$$

where $\ell \in \{1, 2, 3\}$, and α, β and γ are algebraic functions of x and z. Note that Theorems A, 2 and 3 are a source of such identities. Computing the z-derivative of the both sides of the latter equality results in

$$\sum_{n=0}^{\infty} u_n n P_{\ell n}(x) z^n = \gamma_0 F(\alpha) F(\beta) + \gamma_1 F(\alpha) G(\beta) + \gamma_2 G(\alpha) F(\beta),$$

where γ_0, γ_1 and γ_2 are again algebraic functions of x and z. We now take algebraic $x = x_0$ and $z = z_0$, from the convergence domain, in the last equalities such that the corresponding quantities $\alpha = \alpha(x_0, z_0)$ and $\beta = \beta(x_0, z_0)$ are values of the modular function $t(\tau)$ at quadratic irrationalities: $\alpha = t(\tau_0)$, and $\beta = t(\tau_0/N)$ or $1 - t(\tau_0/N)$ for an integer $N > 1$.

Modular equations

Using the corresponding modular equation of degree N, we can always express $F(\beta)$ and $G(\beta)$ by means of $F(\alpha)$ and $G(\alpha)$ only:

$$F(\beta)=\mu_0F(\alpha) \quad \text{and} \quad G(\beta)=\lambda_0F(\alpha)+\lambda_1G(\alpha)+\frac{\lambda_2}{\pi F(\alpha)},$$

where μ_0, λ_0, λ_1, and λ_2 are algebraic (in fact, $\lambda_2=0$ when $\beta=t(\tau_0/N)$). Substituting these relations into the equalities from the previous slide and choosing the algebraic numbers A and B appropriately, we find that the sum $\sum_{n=0}^{\infty}u_n(A+Bn)P_{\ell n}(x_0)z_0^n$ is an algebraic multiple of π,

$$\sum_{n=0}^{\infty}u_n(A+Bn)P_{\ell n}(x_0)z_0^n=\frac{C}{\pi}$$

where A, B and C are algebraic numbers.

Identities for $1/\pi$

In practice, all the algebraic numbers involved are extremely cumbersome, so that the computations happen to be quite involved.
Using the theorems we are able to produce many more examples of the type considered by Sun:

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{n!^2} (2+15n) P_{2n}\left(\frac{3\sqrt{3}}{5}\right)\left(\frac{2\sqrt{2}}{5}\right)^{2n} = \frac{15}{\pi},$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{3})_n(\frac{2}{3})_n}{n!^2}(1+9n)P_{3n}\left(\frac{4}{\sqrt{10}}\right)\left(\frac{1}{3\sqrt{10}}\right)^{3n} = \frac{\sqrt{15+10\sqrt{3}}}{\pi\sqrt{2}},$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n+k}{k}\binom{n}{k}^2 \cdot (16-5\sqrt{10}+60n) \times P_n\left(\frac{5\sqrt{2}+17\sqrt{5}}{45}\right)\left(\frac{5\sqrt{2}-3\sqrt{5}}{5}\right)^n = \frac{135\sqrt{2}+81\sqrt{5}}{\pi\sqrt{2}}.$$