The Life of 1/Pi

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CARMA

π

It is not known exactly where and when π was born. It is at least 25 centuries old and still far from retiring. It has many mathematical faces. For example, a plane geometer can think of it as of the ratio of a circle's circumference to its diameter. To an analyst, it can be thought as part of the famous five-constant expression $e^{\pi\sqrt{-1}} + 1 = 0$ or, better, of Stirling's formula $n! \sim (n/e)^n \sqrt{2\pi n}$ for the asymptotics of the gamma function $n! = \Gamma(n+1)$ as $n \to \infty$. There are definitions of π for group theorists, combinatorialists, maths physicists, probabilists, $\dots$. To a number theorist, the constant is associated with $\zeta(2) = \pi^2/6$ — Euler's famous resolution of *Basel's problem*. The number π also happens to be transcendental, thanks to Lindemann.

As once told by Alf van der Poorten, "in the years BC – before calculators — π was $22/7$ and in the years AD — after decimals — π became $3.141\,592\,65\dots$" The 22nd of July ($22/7$) is recognised as the *Pi Approximation Day*, while March 14 (3.14 in the month.day format) is celebrated annualy as the *Pi Day*.

Computing π

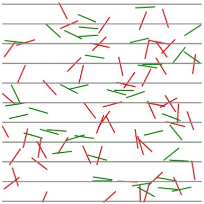
In the 20th and 21st centuries, mathematicians and computer scientists discovered new approaches that, when combined with increasing computational power, extended the decimal representation of π to, as of this year, over 13.3 trillion (1013) digits.

In reality, all scientific applications require no more than a few hundred digits of π, and many substantially fewer, so the primary motivation for these computations is the human desire to break records.

However, the extensive calculations involved have been used to test supercomputers and high-precision multiplication algorithms. A usual practice here is using two or three different algorithms for computing the number. Remarkably, one of the fastest algorithms to do so actually computes $1/\pi$ rather than π itself. And that where the dark side of the constant in question shows up.

$1/\pi$

Suppose we have a floor made of parallel strips of wood, each the same width of 2 inches, and we drop a 1-inch needle onto the floor. What is the probability that the needle will lie across a line between two strips?



The question is known as *Buffon's needle problem*. Georges-Louis Leclerc, Comte de Buffon asked this question in 1733 at the age of 25 and answered it himself only in 1777. The probability is $1/\pi = 0.318\ldots$. This seems to be the first 'natural' appearance of the number in mathematics.

Some time between those two historical events James Cook sailed along the Australian mainland and mapped the east coast, which he named New South Wales and claimed for Great Britain.

Starting this year Newcastle celebrates 31.8 (as suggested by the decimal record of $100/\pi$) as the Pi Down Under Day.

$\rotatebox[origin=c]{180}{π}$

It is not that important, from all practical points of view, to consider $1/\pi$ or c/π, where the coefficient c belongs to a reasonable (and fast computable!) set; for example, $c \in \mathbb{Z}$ or $c \in \mathbb{Q}$ or c is an algebraic number.

In my talk I will deliberately think of $1/\pi$ as of any such c/π, mostly in order to avoid unpleasant (or unnatural) scaling.

In this broader sense we find out many other instances of the pi-down-under, $\rotatebox[origin=c]{180}{π}$, in mathematics; like $1/(2\pi i)$ in complex (analysis) integrals, or $2/\pi$ as the expectation of the two-step length-one uniform random walk (that is, the expected distance from the origin after 2 steps of length 1 each taken into a uniformly random direction). By the way, the expectation of the one-step walk is 1, while the corresponding distance for the three-step walk is more involved (but still closed-formed!):

$$\frac{3 \times 2^{1/3}\Gamma(1/3)^6}{16\pi^4} + \frac{27 \times 2^{2/3}\Gamma(2/3)^6}{4\pi^4}.$$

Hypergeometric function

The analogous expectation for the four-step walk can be expressed in terms of *hypergeometric functions*:

$$\frac{3\pi}{4}\,{}_7F_6\left(\left.\begin{matrix}\frac{7}{4},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{3}{4},2,2,2,1,1\end{matrix}\,\right|\,1\right)-\frac{3\pi}{8}\,{}_7F_6\left(\left.\begin{matrix}\frac{7}{4},\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{3}{4},2,2,2,2,1\end{matrix}\,\right|\,1\right).$$

The latter two explicit evaluations were recently established by J. Borwein, A. Straub and J. Wan (the project I joined at some stage for aesthetical reasons).

The (generalised) hypergeometric function is defined by the series

$${}_mF_{m-1}\left(\left.\begin{matrix}a_1,\,a_2,\,\ldots,\,a_m\\ b_2,\,\ldots,\,b_m\end{matrix}\,\right|\,z\right)=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_m)_n}{(b_2)_n\cdots(b_m)_n}\,\frac{z^n}{n!}$$

in the disc $|z|<1$ and as the analytic continuation of the latter to complex plane (some cuts of the plane are required). The shifted factorial (aka Pochhammer's symbol) $(a)_n$ is defined as $\Gamma(a+n)/\Gamma(a)=a(a+1)(a+2)\cdots(a+n-1)$.

Hypergeometric formulae for $1/\pi$

In 1859, in a long paper in Crelle's Journal, G. Bauer examined certain Fourier series expansions and established the formula

$$\sum_{n=0}^{\infty} \binom{2n}{n}^3 (1+4n)\left(-\frac{1}{16}\right)^n = \frac{2}{\pi}.$$

Proving it is nowadays a nice (but still challenging!) exercise. There is an elementary way of doing it using a telescoping argument (coming from the Gosper–Wilf–Zeilberger theory) but for that one needs to introduce one extra parameter and to apply some classical results from analysis (like Carlson's theorem). There are more advanced proofs based on the theory of complex multiplication on elliptic curves or on modular forms.

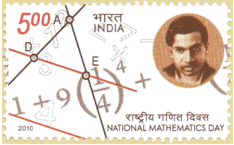
More importantly, the formula, especially written in the hypergeometric form

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{n!^3} (1+4n) \cdot (-1)^n = \frac{2}{\pi},$$

is a simple representative of a (rare!) family of expressions for $1/\pi$ known as Ramanujan-type formulae. (Note the slow convergence!)

Ramanujan in 1914

Brauer's formula features the form of much more efficient series for $1/\pi$ listed by S. Ramanujan in 1914. His list consisted of 17 formulae including the following examples:



$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{n!^3}(1+10n)\frac{1}{3^{4n}} = \frac{9\sqrt{2}}{4\pi},$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{n!^3}(3+40n)\frac{1}{7^{4n}} = \frac{49\sqrt{3}}{9\pi},$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{n!^3}(1103+26390n)\frac{1}{99^{4n}} = \frac{99^2}{2\pi\sqrt{2}}.$$

Ramanujan's comment about the last identity says "The last series [...] is extremely rapidly convergent."

Proofs of Ramanujan's formulae

The identities do not look hard. In spite of this, their first proofs were only obtained in the 1980s by the Borweins and by the Chudnovskys.
The Chudnovskys proved a related identity

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{6})_n(\frac{1}{2})_n(\frac{5}{6})_n}{n!^3}(545140134n + 13591409) \cdot \frac{(-1)^n}{53360^{3n+2}} = \frac{3}{2\pi\sqrt{10005}}\,,$$

which enabled them to hold the record for the calculation of π in 1989–94. It is this formula that is still in use for computing π.
There are already several surveys addressing a historical account of contemporary techniques for proving Ramanujan's and generalised Ramanujan-type formulae as well as their supercongruence relatives.
The major method which, for example, works for *each* formulae from Ramanujan's list is based on modular parameterisations of the underlying hypergeometric series.
This modular technique cannot be counted as elementary, and it does not cover certain important generalisations of Ramanujan's identities recently discovered by J. Guillera (and proved by him in part by the telescoping machinery).

A 2-slide elementary proof. Slide 1

Let me outline an elementary proof of one formula (telescoping-resistant!) from Ramanujan's list:

$$\sum_{n=0}^{\infty} \frac{(\frac12)_n(\frac14)_n(\frac34)_n}{n!^3}\,(3+40n)\,\frac{1}{7^{4n}} = \frac{49\sqrt{3}}{9\pi}.$$

There is a hidden use of modular functions in designing the proof. Use your favourite CAS (computer algebra system) to verify the hypergeometric identity

$${}_3F_2\left(\begin{matrix}\frac12,\ \frac12,\ \frac12\\ 1,\ 1\end{matrix}\,\middle|\, x\right) = r\cdot {}_3F_2\left(\begin{matrix}\frac14,\ \frac12,\ \frac34\\ 1,\ 1\end{matrix}\,\middle|\, y\right)$$

where

$$x = -\frac{4p(1-p)(1+p)^3(2-p)^3}{(1-2p)^6}, \quad y = \frac{16p^3(1-p)^3(1+p)(2-p)(1-2p)^2}{(1-2p+4p^3-2p^4)^4},$$

$$r = \frac{(1-2p)^3}{1-2p+4p^3-2p^4}.$$

Differentiate it.

A 2-slide elementary proof. Slide 2

Differentiate it (a and b are arbitrary):

$$\left(a + bx\frac{\mathrm{d}}{\mathrm{d}x}\right){}_3F_2\left(\left.\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\\ 1,\ 1\end{matrix}\,\right|\, x\right)$$
$$= \left(a + bx\frac{\mathrm{d}r}{\mathrm{d}x} + b\frac{rx}{y}\frac{\mathrm{d}y}{\mathrm{d}x}\cdot y\frac{\mathrm{d}}{\mathrm{d}y}\right)\cdot {}_3F_2\left(\left.\begin{matrix}\frac{1}{4},\ \frac{1}{2},\ \frac{3}{4}\\ 1,\ 1\end{matrix}\,\right|\, y\right).$$

Finally, choose $p = (1 - \sqrt{45 - 18\sqrt{6}})/2$, so that $x = -1$ and $y = 1/7^4$, and $a = 1$, $b = 4$ to recognise the left-hand side as the familiar Bauer's series

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{n!^3}(1 + 4n)\cdot(-1)^n = \frac{2}{\pi}.$$

The right-hand side is then the wanted evaluation

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{n!^3}(3 + 40n)\frac{1}{7^{4n}} = \frac{49\sqrt{3}}{9\pi}.$$

‘Translation’ method

The method outlined above is general enough to be used in other situations for proving *some* other formulae for $1/\pi$. The underlying mechanism is based on existence of suitable *transformation* identities of hypergeometric functions and the chain rule.

The method also allows one to establish identities of the form

$$\sum_{n=0}^{\infty} A_n\,(a+bn)\,z_0^n = \frac{c}{\pi},$$

where A_n is not necessarily a quotient of Pochhammer's symbols but, more generally, a binomial sum.

Such formulae are known as Ramanujan–Sato-type formulae after 2002 T. Sato's discovery of the formula

$$\sum_{n=0}^{\infty} A_n\cdot(10-3\sqrt{5}+20n)\left(\frac{\sqrt{5}-1}{2}\right)^{12n} = \frac{20\sqrt{3}+9\sqrt{15}}{6\pi}$$

of Ramanujan type, involving Apéry's numbers

$$A_n = \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 \in \mathbb{Z}, \qquad n=0,1,2,\dots .$$

Modular machinery

Modular parameterisation of the function $\sum_{n=0}^{\infty} A_n z^n$ is a universal recipe to prove any example of the type

$$\sum_{n=0}^{\infty} A_n \, (a + bn) \, z_0^n = \frac{c}{\pi}.$$

The corresponding methods are developed in the works of H. H. Chan, S. Cooper and their numerous collaborators and followers. They also gathered and listed the so-called rational examples of such formulae, that is, the series in which a, b and z_0 are *rational* numbers, like in Brauer's formula and the three examples above from Ramanujan's list.

Here are some entries:

$$\sum_{n=0}^{\infty} \left\{ \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n} \right\} (71 + 682n) \frac{(-1)^n}{15228^n} = \frac{162\sqrt{47}}{5\pi},$$

$$\sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n} \binom{2k}{n} \right\} (1286 + 11895n) \frac{(-1)^n}{22^{3n}} = \frac{10648}{\pi\sqrt{7}}.$$

Semi-Ramanujan identities

In our recent joint work with Guillera we went on simplifying the existing techniques even further, by avoiding the Wilf–Zeilberger telescoping method (as it works not often enough). The idea is to 'translate' not existing Ramanujan-type series but something much simpler and accessible to classical methods. One family of such 'semi-Ramanujan' identities is as follows.

For s in the interval $0 < s < 1$, the following limit is true:

$$\lim_{x\to 1^-} \sqrt{1-x} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (s)_n (1-s)_n}{n!^3} n x^n = \frac{\sin \pi s}{\pi}.$$

The proof makes use of the Stolz–Cesàro theorem and the limit

$$\lim_{n\to\infty} \frac{(s)_n (1-s)_n n}{n!^2} = \frac{1}{\Gamma(s)\,\Gamma(1-s)} = \frac{\sin \pi s}{\pi}.$$

Other applications

More examples of the machinery:

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{6})_n(\frac{5}{6})_n}{n!^3}(1+11n)\left(\frac{4}{125}\right)^n = \frac{5\sqrt{5}}{2\pi\sqrt{3}},$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{3})_n(\frac{2}{3})_n}{n!^3}(1+5n)\left(-\frac{9}{16}\right)^n = \frac{4}{\pi\sqrt{3}},$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{n!^3}(1+10n)\frac{1}{3^{4n}} = \frac{9\sqrt{2}}{4\pi},$$

as well as

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}^4(1+3n)\left(-\frac{1}{20}\right)^n = \frac{5}{2\pi}$$

and even

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}^4\left(-\frac{1}{20}\right)^n = \frac{\pi}{5^{1/4}\Gamma(\frac{3}{4})^4}.$$

Guillera's generalisations

In the beginning of the 2000s Guillera realised that the Wilf–Zeilberger telescoping machinery together with an intelligent introducing of extra parameter(s) on the right places can be used to prove not only some other identities of Ramanujan but also formulae like

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^5}{n!^5}(20n^2+8n+1)\frac{(-1)^n}{2^{2n}} = \frac{8}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^5}{n!^5}(820n^2+180n+13)\frac{(-1)^n}{2^{10n}} = \frac{128}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3(\frac{1}{4})_n(\frac{3}{4})_n}{n!^5}(120n^2+34n+3)\frac{1}{2^{4n}} = \frac{32}{\pi^2}.$$

He also to find experimentally several additional formulae, some with G. Almkvist.

More about Guillera's formulae for $1/\pi^2$

One of the experimental findings of Guillera and Almkvist is the 'irrational' formula

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3(\frac{1}{3})_n(\frac{2}{3})_n}{n!^5}\big((32-216\phi)n^2+(18-162\phi)n+(3-30\phi)\big)(3\phi)^{3n} \stackrel{?}{=} \frac{3}{\pi^2},$$

where $\phi = \big((\sqrt{5}-1)/2\big)^5 = 0.09016994\dots$.

The differential equations satisfied by (hypergeometric) functions in Guillera's formulae for $1/\pi^2$ have now order 5.
There is no obvious way to deduce any of these newer formulae by modular means; the problem lies in the fact that the (Zariski closure of the) projective monodromy group for the corresponding series $F(z) = \sum_{n=0}^{\infty} A_n z^n$ is always $O_5(\mathbb{R})$ (this is a consequence of a general result of F. Beukers and G. Heckman about the monodromy of hypergeometric differential equations), which is essentially 'richer' than $O_3(\mathbb{R})$ for classical Ramanujan's series.

Calabi–Yau differential equations

It is a challenge to develop a modular-like theory for proving Guillera's identities and finding a (more or less) general pattern of them.
For the moment, there are only speculations about their possible relationship to mirror symmetry, namely, to the linear differential equations for the periods of certain Calabi–Yau three- and fourfolds.
Guillera's WZ algorithmic method is the only proving technique available.

3D and 4D generalisations

There exists also the ‘3D’ identity

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^7}{n!^7}(168n^3 + 76n^2 + 14n + 1)\frac{1}{2^{6n}} \stackrel{?}{=} \frac{32}{\pi^3}$$

discovered by B. Gourevich in 2002 (using an integer relations algorithm), and the more recent formula

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^7(\frac{1}{4})_n(\frac{3}{4})_n}{n!^9\, 2^{12n}}(43680n^4+20632n^3+4340n^2+466n+21) \stackrel{?}{=} \frac{2048}{\pi^4}$$

due to J. Cullen (2010).

Ramanujan-type supercongruences

Another striking arithmetic hidden in Ramanujan's formulae for $1/\pi$ and their generalisations relies on the so-called supercongruences.
It happens that if we truncate the corresponding hypergeometric series at $n = p - 1$, we always get congruences modulo high powers of p, where $p > 3$ is a prime not dividing the denominator of z_0. For example,

$$\sum_{n=0}^{p-1} \frac{(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{n!^3}(20n+3)\frac{(-1)^n}{2^{2n}} \equiv 3\left(\frac{-1}{p}\right)p \pmod{p^3},$$

$$\sum_{n=0}^{p-1} \frac{(\frac{1}{2})_n^3(\frac{1}{4})_n(\frac{3}{4})_n}{n!^5}(120n^2+34n+3)\frac{1}{2^{4n}} \equiv 3p^2 \pmod{p^5},$$

$$\sum_{n=0}^{p-1} \frac{(\frac{1}{2})_n^7(\frac{1}{4})_n(\frac{3}{4})_n}{n!^9\,2^{12n}}(43680n^4+20632n^3+4340n^2+466n+21) \overset{?}{\equiv} 21p^4 \pmod{p^9}$$

The known proofs (due to myself and Guillera) of some instances use the Wilf–Zeilberger theory again.

Some supercongruence consequences

The supercongruences can be extended to general algebraic (not just rational!) settings. They can be associated with divergent

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{(1)_n^3}(3n+1)2^{2n} \text{ “}=\text{” } \frac{-2i}{\pi}, \qquad \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{(1)_n^3}(3n+1)(-1)^n 2^{3n} \text{ “}=\text{” } \frac{1}{\pi}$$

and complex

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{(1)_n^3}\left(\frac{105-21\sqrt{-7}}{32}n+\frac{49-13\sqrt{-7}}{64}\right)\cdot\left(\frac{47+45\sqrt{-7}}{128}\right)^n = \frac{\sqrt{7}}{\pi}$$

formulae for $1/\pi$.

The story also brought the Suns to the subject: the twin brothers Zhi-Hong Sun and Zhi-Wei Sun, whose main speciality is proving congruences.
Interesting enough, the two have only one paper together, “Fibonacci numbers and Fermat's last theorem” (published in 1992 in *Acta Arithmetica*).

Sun's identities

In fact, Zhi-Wei Sun went further to discover experimentally in 2011–12 some remarkable Ramanujan-type looking examples including

$$\sum_{n=0}^{\infty} \binom{2n}{n}^2 (7+30n) \frac{T_n(34,1)}{(-2^{10})^n} = \frac{12}{\pi},$$

$$\sum_{n=0}^{\infty} \frac{(4n)!}{(2n)!n!^2} (13+80n) \frac{T_n(7,4096)}{(-168^2)^n} = \frac{14\sqrt{210}+21\sqrt{42}}{8\pi},$$

$$\sum_{n=0}^{\infty} \binom{2n}{n}^2 (1+10n) \frac{T_{2n}(38,1)}{240^{2n}} = \frac{15\sqrt{6}}{4\pi};$$

$T_n(b,c)$ denotes the coefficient of x^n in the expansion of $(x^2+bx+c)^n$,

$$T_n(b,c) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k} b^{n-2k} c^k.$$

The above instances were subsequently proven in our joint work with H. H. Chan and J. Wan.

Generating functions of Legendre polynomials

The key to proving the identities is the modular parameterisation of the corresponding generating functions. The latter happen to be generating functions of classical Legendre polynomials

$$P_n(x) = \sum_{m=0}^{n} \binom{n}{m}^2 \left(\frac{x-1}{2}\right)^m \left(\frac{x+1}{2}\right)^{n-m} = {}_2F_1\left(\begin{matrix} -n, \ n+1 \\ 1 \end{matrix} \,\middle|\, \frac{1-x}{2}\right),$$

because Sun's binomial sums $T_n(b,c)$ are in essence $P_n(x)$.
For example, we use

$$\sum_{n=0}^{\infty} \frac{(s)_n(1-s)_n}{n!^2} P_n(x) z^n$$
$$= {}_2F_1\left(\begin{matrix} s, \ 1-s \\ 1 \end{matrix} \,\middle|\, \frac{1-\rho-z}{2}\right) \cdot {}_2F_1\left(\begin{matrix} s, \ 1-s \\ 1 \end{matrix} \,\middle|\, \frac{1-\rho+z}{2}\right),$$

where $\rho = \rho(x,z) = (1-2xz+z^2)^{1/2}$, which is the identity of W. N. Brailey and F. Brafman from the 1950s.

New generating functions

In joint joint work with J. Wan in 2011 we also established new generating functions of Legendre polynomials, including

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{n!^2} P_{2n}\left(\frac{(1-X-Y)(X+Y-2XY)}{(Y-X)(1-X-Y+2XY)}\right) \cdot \left(\frac{X-Y}{1-X-Y+2XY}\right)^{2n}$$

$$= (1-X-Y+2XY)\, {}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}\\ 1\end{matrix} \,\middle|\, 4X(1-X)\right) \cdot {}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}\\ 1\end{matrix} \,\middle|\, 4Y(1-Y)\right),$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{3})_n(\frac{2}{3})_n}{n!^2} P_{3n}\left(\frac{(X+Y)(1-X-Y+3XY)-2XY}{(Y-X)\sqrt{p(X,Y)}}\right) \cdot \left(\frac{X-Y}{\sqrt{p(X,Y)}}\right)^{3n}$$

$$= \frac{\sqrt{p(X,Y)}}{(1-3X)(1-3Y)}\, {}_2F_1\left(\begin{matrix}\frac{1}{3}, \frac{2}{3}\\ 1\end{matrix} \,\middle|\, -\frac{9X(1-3X+3X^2)}{(1-3X)^3}\right)$$

$$\times\, {}_2F_1\left(\begin{matrix}\frac{1}{3}, \frac{2}{3}\\ 1\end{matrix} \,\middle|\, -\frac{9Y(1-3Y+3Y^2)}{(1-3Y)^3}\right),$$

where $p(X,Y) = (1-X-Y+3XY)^2 - 4XY$. These are valid in a neighbourhood of $X = Y = 0$.

A general pattern?

In the Legendre-polynomial examples as well as in many other instances on Sun's list it is shown that the generating functions satisfy 4th-order 'arithmetic' differential equations (not 3rd-order, like in the Ramanujan-type formulae) whose monodromy is commensurable with a discrete group of $O_4(\mathbb{R})$.

The corresponding monodromy group can be identified as the tensor of two SL_2's, and this fact leads to representing the solutions (that is, the generating functions) as the product of solutions of two 2nd-order 'arithmetic' DEs.

We expect that this phenomenon happens in general.

There are several supporting factorisations of the 4th-order DEs coming not only from our joint work with Wan but also from the old work of Bailey as well as from some recent investigations of Straub and M. Rogers, Beukers, Straub and myself.

This problem has several consequences not only in number theory and special functions. It initiates the related research in algorithmic theory of linear differential equations.

Summary

The number $1/\pi$ has first shown its 'independence' of π and, at the same time, importance in computing the latter in Ramanujan's hypergeometric identities for $1/\pi$.

The proofs of Ramanujan's formulae initiated research in modular forms and elliptic functions, and later in generalised hypergeometric functions, with links to the special linear differential equations satisfied by the periods of Calabi–Yau manifolds.

More recent discoveries inspired new work in the theory of differential equations, mathematical physics and special polynomials.

There are still many questions to address.

Life of . . . ($\geq$ 2016)

. . . constants that are (expected to be) irrational (and transcendental)

. . . short random walks (*Jon Borwein*) (*Armin Straub*)

. . . Mahler measures

. . . Mahler functions (*Michael Coons*)

. . . (multiple) zeta values

. . . positive rational functions (*Armin Straub*)

. . . hypergeometric functions

. . . Number Theory Down Under Special Interest Group
of the Australian Mathematical Society

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Secretary: Mumtaz Hussain <mumtaz.hussain@newcastle.edu.au>

Long Live $1/\pi$!