Rogers–Ramanujan identities, dilogarithm identities and experimental mathematics

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q-Exponential function

For $z \in \mathbb{C}$, $|z| < 1$, define the q-exponential function

$$e(z) = e_q(z) = \frac{1}{(z)_\infty} = \frac{1}{\prod_{n=0}^\infty (1 - zq^n)}.$$

The similarity with the classical exponential function comes from the expansion

$$e(z) = \sum_{n=0}^\infty \frac{z^n}{(q)_n} = \sum_{n=0}^\infty \frac{z^n}{(1-q)^n [n]_q!},$$

where the q-polynomials $[n]_q! = \frac{(q)_n}{(1-q)^n} = \prod_{k=1}^n \frac{1-q^k}{1-q}$ may be viewed as q-factorials since $[n]_q! \to n!$ as $q \to 1$.

In addition, this function satisfies the "exponential" functional identity

$$e(X + Y) = e(X)e(Y),$$

if $e(X) = e_q(X)$, $e(Y) = e_q(Y)$ and $e(X+Y) = e_q(X+Y)$ are viewed as elements in the Weyl algebra $\mathbb{C}_q[[X, Y]]$ of formal power series in two elements X, Y linked by the commutation relation $XY = qYX$.

Quantum dilogarithm

On the other hand, from the infinite product representation we have the asymptotic behaviour

$$\log e(z) = \sum_{n=0}^{\infty} \bigl(-\log(1-q^n z)\bigr) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{q^{mn} z^m}{m} = \sum_{m=1}^{\infty} \frac{z^m}{m(1-q^m)}$$

$$= \frac{1}{1-q} \sum_{m=1}^{\infty} \frac{z^m}{m(1-q^m)/(1-q)} \sim \frac{-1}{\log q} \sum_{m=1}^{\infty} \frac{z^m}{m^2} \qquad \text{as} \quad q \to 1,$$

already mentioned by S. Ramanujan, since $(1-q^m)/(1-q) \to m$ and $\log q \sim q-1$ as $q \to 1$.
This allows one to call $\log e(z)$ (and, thus, $e(z)$ itself) a q-analogue of the dilogarithm function

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

Quantum pentagonal identity

This analogy is much deeper than just the asymptotics above because it is not hard to check that the q-binomial theorem

$$\sum_{n=0}^{\infty} \frac{(x)_n}{(q)_n} y^n = \frac{(xy)_\infty}{(y)_\infty}$$

is equivalent to the so-called quantum pentagonal identity

$$e(X)e(Y) = e(Y)e(-YX)e(X),$$

where as before $e(X) = e_q(X)$, $e(Y) = e_q(Y)$ and $e(-YX) = e_q(-YX)$ are elements in the Weyl algebra $\mathbb{C}_q[[X, Y]]$.

It seems that B. Richmond and G. Szekeres (1981) were the first to realise that the limiting case $q \to 1$ of certain q-hypergeometric identities (in fact, they considered the Andrews–Gordon generalisation of the Rogers–Ramanujan identities) produces non-trivial identities for the dilogarithm values.

The argument was later exploited by J. Loxton (1984) and rediscovered in the context of the q-binomial theorem and quantum dilogarithm by L. Faddeev and R. Kashaev (1994, 2004).

5-Term identity for the dilogarithm

Theorem 1

The limiting case $q \to 1$ of the q-binomial theorem gives the equality

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(y) = \mathrm{Li}_2\left(\frac{x}{1-y}\right) + \mathrm{Li}_2\left(\frac{y}{1-x}\right) - \mathrm{Li}_2\left(\frac{xy}{(1-x)(1-y)}\right) - \log(1-x)\log(1-y), \qquad 0 < x < 1, \quad 0 < y < 1.$$

Although we prove the 5-term relation for x and y restricted to the interval $(0, 1)$, and this positivity is always crucial in application of the allied asymptotical formulae, the identity remains valid for $x, y \in \mathbb{C} \setminus (1, +\infty)$ by the theory of analytic continuation.
The formula in the theorem is due to N. Abel (the 1820s) but an equivalent formula was published by W. Spence (1809) nearly twenty years earlier. Another equivalent (but "cleaner") form, which we discuss below, was given by L. Rogers (1907).

Proof

Without loss of generality assume that q is sufficiently close to 1, namely, that $\max\{x, y, 1-y(1-x)\} < q < 1$. The easy part of the theorem is the asymptotics of the right-hand side of the q-binomial theorem:

$$\log\frac{(xy)_\infty}{(y)_\infty} = \log\frac{e(y)}{e(xy)} \sim \frac{1}{\log q}\big(\mathrm{Li}_2(xy) - \mathrm{Li}_2(y)\big) \qquad \text{as} \quad q \to 1,$$

For the left-hand side, write

$$\sum_{n=0}^{\infty}\frac{(x)_n}{(q)_n}y^n = \sum_{n=0}^{\infty}c_n, \qquad \text{where} \quad c_n = \frac{(x)_n}{(q)_n}y^n > 0.$$

Then the sequence

$$d_n = \frac{c_{n+1}}{c_n} = \frac{1-xq^n}{1-q^{n+1}}y > 0, \qquad n = 0, 1, 2, \dots,$$

satisfies

$$\frac{d_{n+1}}{d_n} = \frac{(1-xq^{n+1})(1-q^{n+1})}{(1-xq^n)(1-q^{n+2})} < 1 - q^n(1-q)(q-x), \qquad n = 0, 1, 2, \dots$$

(we use $0 < x < q < 1$), hence it is strictly decreasing.

Proof (continued)

On the other hand, $1 - y(1-x) < q$ implies

$$d_0 = \frac{c_1}{c_0} = \frac{1-x}{1-q} y > 1, \quad \text{while} \quad \lim_{n\to\infty} d_n = \lim_{n\to\infty} \frac{1-xq^n}{1-q^{n+1}} y = y < 1.$$

Thus, there exists the unique index $N \ge 1$ such that

$$d_{N-1} = \frac{c_N}{c_{N-1}} \ge 1 \qquad \text{and} \qquad d_N = \frac{c_{N+1}}{c_N} < 1.$$

Solving we obtain N to be the integer part of

$$T = \frac{1}{\log q} \cdot \log \frac{1-y}{q-xy},$$

and some straightforward analysis implies the asymptotical behaviour of the sum $\sum_n c_n$ is fully determined by the asymptotics of its single term c_N:

$$\log \sum_{n=0}^{\infty} c_n \sim \log c_N \sim \log\left(\frac{e(q)e(xq^T)}{e(x)e(q^{T+1})} y^T\right) \qquad \text{as} \quad q \to 1.$$

Finally, we apply the asymptotics of $e(z)$. □

Rogers dilogarithm

The Rogers dilogarithm is a real function defined in the interval $0 < x < 1$ by the formula

$$L(x) = \mathrm{Li}_2(x) + \frac{1}{2}\log x \log(1-x)$$

and then extended to the rest of the real line by setting $L(0) = 0$, $L(1) = \pi^2/6$, and

$$L(x) = 2L(1) - L\left(\frac{1}{x}\right) \quad \text{if } x > 1, \quad \text{and} \quad L(x) = -L\left(\frac{-x}{1-x}\right) \quad \text{if } x < 0.$$

The resulting function is then a monotone increasing continuous real-valued function on $\mathbb{R}$ with limiting values

$$\lim_{x\to-\infty} L(x) = -L(1) - \frac{\pi^2}{6} \quad \text{and} \quad \lim_{x\to+\infty} L(x) = 2L(1) = \frac{\pi^2}{3},$$

and is (real-)analytic except at $x = 0$ and $x = 1$, where its derivative becomes infinite.

5-Term relation revisited

The 5-term relation for this newer version of the dilogarithm reads

$$L(xy) + L\left(\frac{x(1-y)}{1-xy}\right) + L\left(\frac{y(1-x)}{1-xy}\right) = L(x) + L(y);$$

some particular cases of the latter are $L(x) + L(1-x) = L(1)$ and Abel's duplication formula

$$L(x^2) = 2L(x) - 2L\left(\frac{x}{1+x}\right).$$

The proof above and our considerations below naturally impose the restriction $0 \le x \le 1$ on the parameter of the Rogers dilogarithm $L(x)$. Richmond and Szekeres realised that the Rogers dilogarithm is the most appropriate function to express the limiting case $q \to 1$ of q-identities: one never gets additional logarithmic terms.

General framework

The strategy used in the proof of Theorem 1 works for practically every classical summation or transformation formula which involves *positive* terms in the q-hypergeometric sums.

In fact, it is perfect in multi-sum settings as well: one needs then to control the behaviour of c_{n+1}/c_n with respect to *every* summation variable n, not just one as we did in the proof above.

The latter case is related to computation of the asymptotics of so-called Nahm sums which occur in character formulae in Conformal Field Theory.

Multi-sum applications

For example, the Bressoud-type A_2 generalisation of the Rogers–Ramanujan identities due to G. Andrews, A. Schilling and O. Warnaar (1999),

$$\sum_{n_1,n_2,m_1,m_2\ge0}\frac{q^{(n_1^2-n_1m_1+m_1^2)+(n_2^2-n_2m_2+m_2^2)}(q^3;q^3)_{n_2+m_2}}{(q)_{n_1-n_2}(q)_{m_1-m_2}(q^3;q^3)_{n_2}(q^3;q^3)_{m_2}(q)_{n_2+m_2}^2}$$
$$=\frac{(q^3,q^3,q^3,q^6,q^6,q^6,q^9,q^9;q^9)_\infty}{(q)_\infty^3},$$

translates into

$$6L(y)-9L(y^2)-2L(y^3)+L(y^6)=-\frac{2}{3}L(1),$$

where $y=1/\big(2\cos\frac{2\pi}{9}\big)$ is the unique zero of the polynomial y^3-3y^2+1 in the interval $0<y<1$.

This relation was established earlier by Loxton with a help of a different Rogers–Ramanujan-type identity.

Notation

In fact, the underlying dilogarithmic identities can serve as an excellent check of complicated multi-q-sum identities. This was an (experimental) part of our strategy with Ole Warnaar to produce a very general family of Rogers–Ramanujan-type identities for the root system A_{N-1}.

For $1 \le a, b \le N-1$, let C_{ab} be the Cartan integers of the Lie algebra A_{N-1}: $C_{aa} = 2$, $C_{a,a\pm1} = -1$ and $C_{ab} = 0$ otherwise.

In what follows $\boldsymbol{\rho} = (\frac{1}{2}, \frac{3}{2}, \dots, n - \frac{1}{2})$, and for $\mathbf{v} = (v_1, \dots, v_n) \in (\mathbb{Z}/2)^n$ we set $|\mathbf{v}| = v_1 + \cdots + v_n$ and $\|\mathbf{v}\|^2 = v_1^2 + \cdots + v_n^2$.

General conjecture ($N = 2n$)

Conjecture 1 (O. Warnaar & Z.)

For k, n positive integers, $N = 2n$,

$$\sum_{\mathbf{M}} \frac{q^{\frac{1}{2}\sum_{a,b=1}^{N-1}\sum_{i=1}^{k-1} C_{ab} M_i^{(a)} M_i^{(b)}}}{\prod_{a=1}^{N-1}\prod_{i=1}^{k-1}(q)_{M_i^{(a)}-M_{i+1}^{(a)}}}$$

$$= \frac{1}{(q)_\infty^{2n^2-n}} \sum_{\mathbf{v}} \prod_{1\le i<j\le n} \frac{v_i^2 - v_j^2}{\rho_i^2 - \rho_j^2} (-1)^{|\mathbf{v}|-|\boldsymbol{\rho}|} q^{(\|\mathbf{v}\|^2-\|\boldsymbol{\rho}\|^2)/(2(2k+2n-1))},$$

where the sum on the left is over $(N-1)(k-1)$-tuples $\mathbf{M} = (M_i^{(a)})_{1\le a\le N-1;\,1\le i\le k-1}$ with non-negative integer values and $M_k^{(a)}$ is set to be 0 for $a = 1, \dots, N-1$, while the sum on the right is over $\mathbf{v} \in (\mathbb{Z}/2)^n$ such that $v_i \equiv \rho_i \pmod{2k+2n-1}$.

Comments

Conjecture 1 links an eta identity of Macdonald to the Rogers–Ramanujan and Andrews–Gordon identities. More specifically, our family of q-series identities depending on positive integers k and n is such that

1. For $k = 1$ we recover Macdonald's $\mathrm{A}_{2n}^{(2)}$ identity for the Dedekind η-function.
2. For $n = 1$ and $k = 2$ we recover, modulo the Jacobi triple product identity, the Rogers–Ramanujan identities.
3. For $n = 1$ and general k we recover instances of the Andrews–Gordon identities.
4. For general n and $k \to \infty$ we recover the A_{2n-1} case of an identity of Hua related to representations of quivers.

There is an $N = 2n - 1$ counterpart of the conjecture established by Feigin and Stoyanovsky (1994, 1998) using a representation theoretic interpretation.

In our work with Warnaar we establish the conjecture for $k = 2$ and arbitrary n using Milne's C_n analogue of the Rogers–Selberg identity.

Kirillov's identity

An interesting fact is that the limiting case $q \to 1$ of Conjecture 1 is the dilogarithmic identity

$$\sum_{a=1}^{N-1} \sum_{i=1}^{k-1} L\left(\frac{\sin\left(\frac{a\pi}{2k+N-1}\right) \sin\left(\frac{(N-a)\pi}{2k+N-1}\right)}{\sin\left(\frac{(i+a)\pi}{2k+N-1}\right) \sin\left(\frac{(i+N-a)\pi}{2k+N-1}\right)} \right) = \frac{N(N-1)(k-1)}{2k+N-1} L(1),$$

which was previously proven by A. Kirillov (1995).

Quantum vs. dilogarithm

It is an interesting task to prove (experimentally) known identities for the values of Rogers dilogarithm at $x \in [0,1] \cap \overline{\mathbb{Q}}$ as limiting cases of certain q-series identities.
For example, the companion

$$6L(z)-9L(z^2)-2L(z^3)+L(z^6)=\frac{2}{3}L(1), \quad z:0<z<1,\ z^3-3z+1=0,$$

to (already mentioned) Loxton's identity

$$6L(y)-9L(y^2)-2L(y^3)+L(y^6)=-\frac{2}{3}L(1), \quad y:0<y<1,\ y^3-3y^2+1=0,$$

remains unproven by the q-asymptotics techniques.
The former identity was conjectured by L. Lewin (there $z=1-y=2\cos\frac{4\pi}{9}$). It was proved by H. Gangl (1993) by iterative use of the 5-term relation.

It is also interesting to record the dilogarithmic consequences of known summation and transformation formulae.

Merci

Thank you!