# HYPERGEOMETRIC (SUPER)CONGRUENCES

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Abstract. The sequence of (terminating balanced) hypergeometric sums

$$a_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2, \qquad n=0,1,\dots,$$

appears in Apéry's proof of the irrationality of $\zeta(3)$. Another example of hypergeometric use in irrationality problems is Ramanujan-type identities for $1/\pi$, like

$$\sum_{k=0}^{\infty}\binom{2k}{k}^3(4k+1)\frac{(-1)^k}{2^{6k}}=\frac{2}{\pi}.$$

These two, seemingly unrelated but both beautiful enough, hypergeometric series have many issues in common, as I explain in my review "Ramanujan-type formulae for $1/\pi$. A second wind?". In my talk, I plan to discuss further number-theoretical aspects of the two examples, namely, the congruences

$$a_{np}\equiv a_n \pmod{p^3} \qquad \text{for} \quad n=0,1,\dots, \quad p>3 \text{ prime}$$

(I. Gessel, 1982), and

$$\sum_{k=0}^{p-1}\binom{2k}{k}^3(4k+1)\frac{(-1)^k}{2^{6k}}\equiv(-1)^{(p-1)/2}p \pmod{p^3} \qquad \text{for} \quad p>2 \text{ prime}$$

(E. Mortenson, 2008); these are called *supercongruences* because they happen to hold not just modulo a prime $p$ but a higher power of $p$. In spite of the elementary character of these supercongruences, the existing proofs are not general enough to treat other similar cases. My goal is to attract attention to this nice and elementary subject on the border of arithmetic and hypergeometrics.

## 1. Hypergeometric series

The (generalized) hypergeometric function is defined by the series

$${}_mF_{m-1}\left(\begin{matrix}a_1, & a_2, & \dots, & a_m\\ & b_2, & \dots, & b_m\end{matrix}\,\middle|\, z\right)=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_m)_n}{(b_2)_n\cdots(b_m)_n}\,\frac{z^n}{n!}, \tag{1}$$

which has the unit disc $|z|<1$ as natural domain of convergence, where

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}=\begin{cases}a(a+1)\cdots(a+n-1) & \text{if } n\ge 1,\\ 1 & \text{if } n=0,\end{cases}$$

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denotes the Pochhammer symbol (the rising factorial). It is a standard exercise to check that the function (1) satisfies the hypergeometric differential equation

$$\biggl(\theta\prod_{j=2}^{m}(\theta+b_j-1)-z\prod_{j=1}^{m}(\theta+a_j)\biggr)y=0, \qquad \theta=z\frac{\mathrm{d}}{\mathrm{d}z}, \tag{2}$$

of order $m$. For a comprehensive knowledge about the hypergeometric functions I refer to the classical books of W. N. Bailey (1935) and L. J. Slater (1966), as well as to the $q$-Bible of G. Gasper and M. Rahman (2004) and to the treatise on special functions of G. E. Andrews, R. Askey and R. Roy (1999). The hypergeometric functions possess many functional equations, known as summation and transformation theorems, sometimes quite unexpected and highly non-trivial. An example (having some role in my talk) is Clausen's identity

$${}_2F_1\biggl(\begin{matrix} a,\, b \\ a+b+\frac12 \end{matrix}\biggm| z\biggr)^2 = {}_3F_2\biggl(\begin{matrix} 2a,\, 2b,\, a+b \\ a+b+\frac12,\, 2a+2b \end{matrix}\biggm| z\biggr). \tag{3}$$

An important feature of the hypergeometrics, at least from the arithmetic point of view, is that many mathematical constants are special values of the hypergeometric functions. This naturally leads one to single out a class of arithmetic hypergeometric series — something that I can hardly formalize but on which I would like to talk further.

Instances of such arithmetic hypergeometric series are those which may be parameterized by modular functions; in other words, under a suitable choice of modular function $z=z(\tau)$, the hypergeometric function ${}_mF_{m-1}$ becomes a modular form (of weight $m-1$). One classical example, due to C. Jacobi, is

$${}_2F_1\biggl(\begin{matrix} \frac12,\, \frac12 \\ 1 \end{matrix}\biggm| \frac{\theta_2^4}{\theta_3^4}\biggr) = \theta_3^2,$$

where

$$\theta_2(\tau)=\sum_{n\in\mathbb{Z}}e^{\pi i\tau(n+1/2)^2} \quad \text{and} \quad \theta_3(\tau)=\sum_{n\in\mathbb{Z}}e^{\pi i\tau n^2}$$

are modular forms of weight $1/2$ with respect to the congruence subgroup $\Gamma(2)$ of $SL_2(\mathbb{Z})$. Another example, due to R. Fricke, is the identity

$${}_2F_1\biggl(\begin{matrix} \frac1{12},\, \frac5{12} \\ 1 \end{matrix}\biggm| \frac{E_4^3-E_6^2}{E_4^3}\biggr)^4 = {}_3F_2\biggl(\begin{matrix} \frac16,\, \frac12,\, \frac56 \\ 1,\, 1 \end{matrix}\biggm| \frac{E_4^3-E_6^2}{E_4^3}\biggr)^2 = E_4, \tag{4}$$

where the two functions

$$E_4(\tau)=1+240\sum_{n=1}^{\infty}\sigma_3(n)e^{2\pi in\tau} \quad \text{and} \quad E_6(\tau)=1-504\sum_{n=1}^{\infty}\sigma_5(n)e^{2\pi in\tau},$$

$\sigma_k(n)=\sum_{d|n}d^k$, generate the algebra of modular forms for the full modular group $SL_2(\mathbb{Z})$.

## 2. Ramanujan's series for $1/\pi$

In 1914 S. Ramanujan recorded a list of 17 (hypergeometric) series for $1/\pi$. One of these series

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{n!^3}(4n+1)\cdot(-1)^n = \frac{2}{\pi} \tag{5}$$

was actually proven by G. Bauer already in 1859; the proof exploits continued fractions and orthogonal polynomials. More sophisticated and quite impressive examples due to Ramanujan are

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{n!^3}(21460n+1123)\cdot\frac{(-1)^n}{882^{2n+1}} = \frac{4}{\pi}\,, \tag{6}$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{n!^3}(26390n+1103)\cdot\frac{1}{99^{4n+2}} = \frac{1}{2\pi\sqrt{2}}, \tag{7}$$

since they produce rapidly converging (rational) approximations to $\pi$ (and $\pi\sqrt{2}$). The Pochhammer products occurring in all formulae of this type may be written purely in terms of binomial coefficients:

$$\begin{aligned} \frac{(\frac{1}{2})_n^3}{n!^3} &= 2^{-6n}\binom{2n}{n}^3, & \frac{(\frac{1}{3})_n(\frac{1}{2})_n(\frac{2}{3})_n}{n!^3} &= 2^{-2n}3^{-3n}\binom{2n}{n}\frac{(3n)!}{n!^3}, \\ \frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{n!^3} &= 2^{-8n}\frac{(4n)!}{n!^4}, & \frac{(\frac{1}{6})_n(\frac{1}{2})_n(\frac{5}{6})_n}{n!^3} &= 12^{-3n}\frac{(6n)!}{n!^3(3n)!}. \end{aligned} \tag{8}$$

Ramanujan's original list was subsequently extended to several other series. The famous series of the Chudnovskys (1989),

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{6})_n(\frac{1}{2})_n(\frac{5}{6})_n}{n!^3}(545140134n+13591409)\cdot\frac{(-1)^n}{53360^{3n+2}} = \frac{3}{2\pi\sqrt{10005}}\,, \tag{9}$$

gives, roughly speaking, 14 decimal (or 47 binary) digits per term. On the left-hand side of each formula (5)–(7), (9) we have linear combinations of a ${}_3F_2$ hypergeometric series (1) and its derivative at a point close to the origin. The rapid convergence of the series in (6), (7), (9) may be used for proving the quantitative irrationality of the numbers $\pi\sqrt{d}$ with $d \in \mathbb{N}$.

It is a matter of taste to decide whether these series are beautiful or not. Nevertheless, their origin and applications make them of definite interest and clear attraction to hypergeometric and computational people as well as to number theorists and those working in the (elliptic) modular business (compare (9) with (4)).

To make the long story shorter (but not really short), I now sketch a hypergeometric proof of the 'worse' formula (5) and then switch to another hypergeometric-modular-computational-number theoretic issue.

## 3. Creative telescoping

An important part of the contemporary theory of hypergeometric series is algorithmic, and one of the bestsellers on the market is the algorithm of creative telescoping due to R. W. Gosper and D. Zeilberger. In 1994, D. Zeilberger and his automatic collaborator S. B. Ekhad demonstrated how one could use the machinery to prove the Bauer–Ramanujan identity (5). One verifies the (terminating) identity

$$\sum_{n=0}^{\infty}\frac{(1/2)_n^2(-k)_n}{n!^2(3/2+k)_n}(4n+1)(-1)^n=\frac{\Gamma(3/2+k)}{\Gamma(3/2)\Gamma(1+k)} \tag{10}$$

for all *non-negative* integers $k$. To do this, divide both sides of (10) by the right-hand side and denote the summand on the left by $F(n,k)$:

$$F(n,k)=(4n+1)(-1)^n\frac{(1/2)_n^2(-k)_n}{n!^2(3/2+k)_n}\,\frac{\Gamma(3/2)\Gamma(1+k)}{\Gamma(3/2+k)};$$

then take

$$G(n,k)=\frac{(2n+1)^2}{(2n+2k+3)(4n+1)}F(n,k)$$

with the motive that $F(n,k+1)-F(n,k)=G(n,k)-G(n-1,k)$, hence $\sum_n F(n,k)$ is a constant, which is seen to be 1 by plugging in $k=0$. Finally, to deduce (5) one takes $k=-1/2$, which is legitimate in view of Carlson's theorem.

Unfortunately, the above argument does not work in general, and it took a while to see that it can be applied to other Ramanujan's and Ramanujan-type identities for $1/\pi$. This was done by J. Guillera (2002–06), who used the method to prove some other identities in several cases when $z$ involves only 2 and 3 in its prime decomposition. In spite of the very narrow applicability, the method allowed Guillera to prove some new generalizations of Ramanujan-type series, namely,

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n^5}{n!^5}(20n^2+8n+1)\frac{(-1)^n}{2^{2n}}=\frac{8}{\pi^2}, \tag{11}$$

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n^5}{n!^5}(820n^2+180n+13)\frac{(-1)^n}{2^{10n}}=\frac{128}{\pi^2}, \tag{12}$$

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n^3(\frac14)_n(\frac34)_n}{n!^5}(120n^2+34n+3)\frac{1}{2^{4n}}=\frac{32}{\pi^2}; \tag{13}$$

for these identities no other proofs are known. In fact, Guillera went further to find experimentally (using an integer relations algorithm) four similar formulae for $1/\pi^2$, and there is known one conjectural ${}_7F_6$ formula due to B. Gourevich (2002) for $1/\pi^3$. The series involved belong to the class of arithmetic hypergeometric functions, although no modular parametrization is known for them.

## 4. Apéry's proof of the irrationality of $\zeta(3)$

In 1978 R. Apéry showed that $\zeta(3)$ is irrational. His rational approximations to the number in question (known nowadays as *Apéry's constant*) have the form $b_n/a_n\in\mathbb{Q}$ for $n=0,1,2,\dots$, where both the denominators

$$a_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2, \qquad n=0,1,2,\dots, \tag{14}$$

and numerators $b_n$ form solutions to the polynomial recurrence

$$(n+1)^3 a_{n+1} - (2n+1)(17n^2+17n+5)a_n + n^3 a_{n-1} = 0 \qquad (15)$$

with the initial data

$$a_0 = 1, \quad a_1 = 5, \qquad b_0 = 0, \quad b_1 = 6.$$

(Highly non-trivial in 1978, this is now a routine exercise using the algorithm of creative telescoping.) According to (14) the numbers $a_n$ are integers. A delicate arithmetic analysis shows that $b_n \cdot d_n^3$ are integers as well, where $d_n$ denotes the least common multiple of $1, 2, \dots, n$. The prime number theorem yields $d_n^{1/n} \to e$ as $n \to \infty$, while the asymptotic behaviour of the approximations is given by $(a_n\zeta(3) - b_n)^{1/n} \to (\sqrt{2}-1)^4$. Assuming that $\zeta(3)$ is rational, say $p/q$, we get the sequence of *positive* integers $(a_n\zeta(3)-b_n)\cdot q d_n^3$ growing like $\big((\sqrt{2}-1)^4 e^3\big)^n < 0.9^n$ — a contradiction for $n$ sufficiently large.

It became an 'expected surprise' in 2002, after T. Sato gave the formula

$$\sum_{n=0}^{\infty} a_n \cdot (20n + 10 - 3\sqrt{5})\left(\frac{\sqrt{5}-1}{2}\right)^{12n} = \frac{20\sqrt{3}+9\sqrt{15}}{6\pi}\,, \qquad (16)$$

that Ramanujan's list can be considerably extended by replacing hypergeometric series on the left-hand sides of (5)–(7), (9) with generating series involving Apéry-like numbers. A uniform method to prove identities like (16), for a not necessarily hypergeometric series $F(z) = \sum_{n=0}^{\infty} a_n z^n$, was initiated and developed in a series of works by H. H. Chan and his collaborators (including myself). The essential ingredient of the method is the fact that $F(z)$ admits a modular parametrization: $f(\tau) = F(z(\tau))$ is a modular form of weight 2 for a suitable modular substitution $z = z(\tau)$. For Apéry's sequence, the parametrization

$$z(\tau) = \left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\right)^{12}, \qquad f(\tau) = \frac{\eta(2\tau)^7\eta(3\tau)^7}{\eta(\tau)^5\eta(6\tau)^5}$$

by modular forms of level 6, was given by F. Beukers in 1985 in his proof of Apéry's theorem using modular forms. Here

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1 - e^{2\pi i n\tau})$$

denotes the Dedekind eta-function.

Other examples of Ramanujan–Sato-type formulae are

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2n-2k}{n-k}\cdot(5n+1)\frac{1}{64^n} = \frac{8}{\pi\sqrt{3}} \qquad (17)$$

due to H. H. Chan, S. H. Chan and Z.-G. Liu (2004);

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/3]}(-1)^{n-k}3^{n-3k}\frac{(3k)!}{k!^3}\binom{n}{3k}\binom{n+k}{k}\cdot(4n+1)\frac{1}{81^n} = \frac{3\sqrt{3}}{2\pi} \qquad (18)$$

due to H. H. Chan and H. Verrill (2005);

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}^4\cdot(4n+1)\frac{1}{36^n}=\frac{18}{\pi\sqrt{15}} \tag{19}$$

due to Y. Yang (2005).

## 5. Supercongruences for Apéry-like numbers

A lot of work was done after Apéry's proof of the irrationality of $\zeta(3)$, to realize what is special in the sequence (14). In 1980, S. Chowla, J. Cowles and M. Cowles observed the supercongruence

$$a_p \equiv a_1 \pmod{p^3} \quad \text{for } p \ge 5, \tag{20}$$

which was shortly after proven by I. Gessel (1982), who established the stronger result

$$a_{np} \equiv a_n \pmod{p^3} \quad \text{for } p \ge 5, \tag{21}$$

valid for all non-negative integers $n$. The name *supercongruences* for the above $p$-adic congruences is because they happen to hold not just modulo a prime $p$ but a higher power of $p$. H. H. Chan, S. Cooper and F. Sica (2009) show that the original method of Gessel can be extended to the other Apéry-like sequences just mentioned, namely, to the sequence of Domb numbers

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2n-2k}{n-k}$$

(cf. (17)) and the binomial sum sequence

$$\sum_{k=0}^{n}\binom{n}{k}^4$$

(cf. (19)). Let me give details for (21) with the choice $a_n=\sum_{k=0}^{n}\binom{n}{k}^4$; the Apéry-like recurrence

$$(n+1)^3a_{n+1}-2(2n+1)(3n^2+3n+1)a_n-4n(4n-1)(4n+1)a_{n-1}=0$$

is due to J. Franel (1894), although it has been rediscovered since then by many others.

Write

$$a_{np}=\sum_{k=0}^{np}\binom{np}{k}^4=\sum_{m=0}^{n}\binom{np}{mp}^4+\sum_{m=0}^{n-1}\sum_{j=1}^{p-1}\binom{np}{mp+j}^4. \tag{22}$$

For each term of the first sum on the right-side of (22) we apply the (super)congruence

$$\binom{np}{mp}\equiv\binom{n}{m} \pmod{p^3} \quad \text{for } p \ge 5$$

due to G. S. Kazandzidis (1968) to arrive at

$$\sum_{m=0}^{n}\binom{np}{mp}^4\equiv\sum_{m=0}^{n}\binom{n}{m}^4=a_n \pmod{p^3}.$$

As for the terms of the second sum in (22), we use the congruence

$$\binom{ap+b}{cp+d} \equiv \binom{a}{c}\binom{b}{d} \pmod p$$

due to E. Lucas (1878) with $a = n-1$, $b = p$, $c = m$ and $d = j$:

$$\binom{np}{mp+j} \equiv \binom{n-1}{m}\binom{p}{j} \pmod p;$$

in addition, note that $\binom{p}{j}$ is divisible by $p$ for each $j = 1, \dots, p-1$, hence the right-hand side is $0 \pmod p$ and, finally, $\binom{np}{mp+j}^4$ is divisible by $p^4$ implying

$$\sum_{m=0}^{n-1}\sum_{j=1}^{p-1}\binom{np}{mp+j}^4 \equiv 0 \pmod{p^4}.$$

Substituting the two resulting congruences into (22) we obtain the validity of (21). As for the congruence (20), even more can be shown, namely, that $a_p \equiv a_1 = 2 \pmod{p^5}$ for primes $p \ge 7$.

It is interesting to note that the above approach is not universal at all: it fails to prove the (experimentally observed) supercongruence (21) for the so-called AZ sequence

$$\sum_{k=0}^{[n/3]}(-1)^{n-k}3^{n-3k}\frac{(3k)!}{k!^3}\binom{n}{3k}\binom{n+k}{k} = \sum_{k=0}^{n}(-1)^{n-k}3^{3(n-k)}\frac{(4k)!}{k!^4}\binom{n+3k}{4k}$$

(cf. (18)). Let me mention that Chan, Cooper and Sica (2009) indicate other conjectural families of supercongruences arising for the sequences of 'modular origin'; examples are

$$c_{np} \equiv c_n \pmod{p^2} \quad \text{for primes } p \equiv 1 \pmod 4, \qquad \text{where} \quad c_n = \frac{(1/4)_n^2}{n!^2}64^n,$$

and

$$c'_{np} \equiv c'_n \pmod{p^2} \quad \text{for primes } p \equiv 1 \pmod 3, \qquad \text{where} \quad c'_n = \frac{(1/6)_n(1/3)_n}{n!^2}108^n.$$

On the other hand, the coefficients of the squares of the generating series,

$$\biggl(\sum_{n=0}^{\infty}c_nz^n\biggr)^2 = \sum_{n=0}^{\infty}\frac{(4n)!}{n!^4}z^n \quad \text{and} \quad \biggl(\sum_{n=0}^{\infty}c'_nz^n\biggr)^2 = \sum_{n=0}^{\infty}\binom{2n}{n}\frac{(3n)!}{n!^3}z^n$$

(the binomial products appear already in (8)), satisfy the stronger congruence (21) from the above-mentioned result of Kazandzidis; in the squaring we use Clausen's formula (3). The problem of developing a method which would work for all binomial sums satisfying Apéry-like recurrences remains open.

## 6. Ramanujan-type supercongruences

It is quite special to me to report in Belgium on another piece of the supercongruence mosaic, which was born in the 1990s in this country. L. Van Hamme (1997) observed that several Ramanujan's and Ramanujan-like formulae for $1/\pi$ (and for some gamma function ratios) admit very nice $p$-adic analogues. I was surprised and impressed last year to follow Van Hamme's pattern and check (for primes up to 1000) the validity of the following supercongruences (the first one comes from Van Hamme's list):

$$\sum_{n=0}^{p-1}\frac{(\frac12)_n^3}{n!^3}(4n+1)(-1)^n \equiv \biggl(\frac{-1}{p}\biggr)p \pmod{p^3} \quad \text{for } p>2, \tag{23}$$

$$\sum_{n=0}^{p-1}\frac{(\frac12)_n(\frac14)_n(\frac34)_n}{n!^3}(21460n+1123)\frac{(-1)^n}{882^{2n}} \equiv 1123\biggl(\frac{-1}{p}\biggr)p \pmod{p^3} \quad \text{for } p>7, \tag{24}$$

$$\sum_{n=0}^{p-1}\frac{(\frac12)_n(\frac14)_n(\frac34)_n}{n!^3}(26390n+1103)\frac{1}{99^{4n}} \equiv 1103\biggl(\frac{-2}{p}\biggr)p \pmod{p^3} \quad \text{for } p>11 \tag{25}$$

(compare them with (5)–(7)), as well as of other examples originated from known Ramanujan-type hypergeometric identities for $1/\pi$. Here $\bigl(\frac{\cdot}{p}\bigr)$ denotes the Legendre symbol. My attention to this was attracted by E. Mortenson and his paper (2008), in which he proved the supercongruence (23). Mortenson's proof makes use of transformation and summation theorems for (terminating) ${}_3F_2$ hypergeometric series — the technique borrowed from an earlier contribution of D. McCarthy and R. Osburn (published however, later, in 2009). It is very hard to believe in strong potentials of this argument in proving other congruences, since the prototype of (23) is the Bauer–Ramanujan identity (5) admitting a simple hypergeometric proof which seems to be non-extendible to other Ramanujan's identities. On the other hand, I show in my recent work that the Zeilberger–Guillera method not only leads to a much simpler proof of (23) but also allows us to prove two more supercongruences of this type:

$$\sum_{n=0}^{p-1}\frac{(\frac12)_n(\frac14)_n(\frac34)_n}{n!^3}(20n+3)\frac{(-1)^n}{2^{2n}} \equiv 3\biggl(\frac{-1}{p}\biggr)p \pmod{p^3} \quad \text{for } p>2 \tag{26}$$

and

$$\sum_{n=0}^{p-1}\frac{(\frac12)_n^3(\frac14)_n(\frac34)_n}{n!^5}(120n^2+34n+3)\frac{1}{2^{4n}} \equiv 3p^2 \pmod{p^5} \quad \text{for } p>2 \tag{27}$$

(cf. (13)). It is remarkable that Guillera's formulae for $1/\pi^2$ transform to congruences $\pmod{p^5}$, while Gourevich's formula for $1/\pi^3$ becomes a congruence $\pmod{p^7}$. Besides the three cases (23), (26) and (27), no proof is known for other $p$-adic Ramanujan-type supercongruences. A lengthy list of these conjectural congruences is given in my recent publication (2009).

At the end of my talk I would like to explain how the algorithm of creative telescoping is used in proving the supercongruences. I do it for (23) to have a parallel with the already-considered proof of (5). First of all note that

$$\sum_{n=0}^{p-1}\frac{(\frac12)_n^3}{n!^3}(4n+1)(-1)^n\equiv\sum_{n=0}^{(p-1)/2}\frac{(\frac12)_n^3}{n!^3}(4n+1)(-1)^n \pmod{p^3},$$

since the Pochhammer products $(\frac12)_n/n!$ are divisible by $p$ for $n>p/2$. For the proof, I slightly re-normalize the WZ-pair from Section 3:

$$F(n,k)=(-1)^{n+k}(4n+1)\frac{(\frac12)_n^2(\frac12)_{n+k}}{(1)_n^2(1)_{n-k}(\frac12)_k^2},$$

$$G(n,k)=-\frac{(2n-1)^2}{2(n-k)(4n-3)}F(n-1,k)=(-1)^{n+k}\cdot 2\cdot\frac{(\frac12)_n^2(\frac12)_{n+k-1}}{(1)_{n-1}^2(1)_{n-k}(\frac12)_k^2};$$

these functions satisfy

$$F(n,k-1)-F(n,k)=G(n+1,k)-G(n,k). \tag{28}$$

Summing (28) over $n=0,1,\dots,\frac{p-1}{2}$, we obtain

$$\sum_{n=0}^{(p-1)/2}F(n,k-1)-\sum_{n=0}^{(p-1)/2}F(n,k)=G(\tfrac{p+1}{2},k)-G(0,k)=G(\tfrac{p+1}{2},k). \tag{29}$$

Furthermore, for $k=1,2,\dots,\frac{p-1}{2}$ we have

$$\begin{aligned}G(\tfrac{p+1}{2},k)&=(-1)^{(p+1)/2+k}\cdot 2\cdot\left(\frac{(\frac12)_{(p-1)/2}}{(1)_{(p-1)/2}}\right)^2\frac{(\frac{p-1}{2}+\frac12)^2(\frac12)_{(p+1)/2+k-1}}{(1)_{(p+1)/2-k}(\frac12)_k^2}\\&=(-1)^{(p+1)/2+k}\cdot 2^{-p}\binom{p-1}{\frac{p-1}{2}}^2p^2\cdot\frac{(\frac12)_{(p+1)/2+k-1}}{(1)_{(p+1)/2-k}(\frac12)_k^2}\equiv 0 \pmod{p^3},\end{aligned}$$

since $(\frac12)_{(p+1)/2+k-1}$ is divisible by $(\frac12)_{(p+1)/2}$, hence by $p$, while the denominator is coprime to $p$. Comparing this result with (29) we see that

$$\sum_{n=0}^{(p-1)/2}F(n,0)\equiv\sum_{n=0}^{(p-1)/2}F(n,1)\equiv\sum_{n=0}^{(p-1)/2}F(n,2)\equiv\dots\equiv\sum_{n=0}^{(p-1)/2}F(n,\tfrac{p-1}{2}) \pmod{p^3}. \tag{30}$$

On the other hand,

$$\begin{aligned}\sum_{n=0}^{(p-1)/2}F(n,\tfrac{p-1}{2})&=F(\tfrac{p-1}{2},\tfrac{p-1}{2})=(4\cdot\tfrac{p-1}{2}+1)\frac{(\frac12)_{p-1}}{(1)_{(p-1)/2}^2}\\&=2\frac{(\frac12)_p}{(1)_{(p-1)/2}^2}=p\cdot 2^{-2(p-1)}\binom{2p-1}{p-1}\binom{p-1}{\frac{p-1}{2}}.\end{aligned} \tag{31}$$

It remains to use the well-known congruences

$$\binom{2p-1}{p-1}\equiv 1 \pmod{p^3}$$

due to J. Wolstenholme (1862) and

$$\binom{p-1}{\frac{p-1}{2}} \equiv (-1)^{(p-1)/2} 2^{2(p-1)} \pmod{p^3} \tag{32}$$

due to F. Morley (1895), although they are only true modulo $p^2$ if $p = 3$, to conclude that the expression in (31) is congruent to

$$\sum_{n=0}^{(p-1)/2} F(n, \tfrac{p-1}{2}) \equiv (-1)^{(p-1)/2} p \pmod{p^4}$$

(for our purposes we need the latter modulo $p^3$), and the desired result follows from (30).

In all proofs of the above congruences, a crucial role is played by auxiliary congruences for the binomial coefficients. The main problem seems to be a purely hypergeometric reduction of complex binomial expressions to the basic ones, and this is something wanted in these nice arithmetic applications of the hypergeometrics.

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