

Ramanujan-type formulae for $1/\pi$ and Legendre polynomials

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Ramanujan's legacy

In 1914 Srinivasa Ramanujan recorded a list of 17 series for $1/\pi$, in particular,

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n (\frac{1}{2})_n (\frac{3}{4})_n}{n!^3} (21460n + 1123) \cdot \frac{(-1)^n}{882^{2n+1}} = \frac{4}{\pi},$$
$$\sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n (\frac{1}{2})_n (\frac{3}{4})_n}{n!^3} (26390n + 1103) \cdot \frac{1}{99^{4n+2}} = \frac{1}{2\pi\sqrt{2}}$$

which produce rapidly converging (rational) approximations to π . Here

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1)\cdots(a+n-1) & \text{for } n \geq 1, \\ 1 & \text{for } n = 0, \end{cases}$$

denotes the Pochhammer symbol (the rising factorial).

Generalisations

An example is the Chudnovskys' famous formula which enabled them to hold the record for the calculation of π in 1989–94:

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{6})_n (\frac{1}{2})_n (\frac{5}{6})_n}{n!^3} (545140134n + 13591409) \cdot \frac{(-1)^n}{53360^{3n+2}} = \frac{3}{2\pi\sqrt{10005}}.$$

A more sophisticated example (which also shows that modularity rather than hypergeometrics is crucial) is Takeshi Sato's formula (2002)

$$\sum_{n=0}^{\infty} u_n \cdot (20n + 10 - 3\sqrt{5}) \left(\frac{\sqrt{5} - 1}{2}\right)^{12n} = \frac{20\sqrt{3} + 9\sqrt{15}}{6\pi}$$

of Ramanujan type, involving Apéry's numbers

$$u_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 \in \mathbb{Z}, \quad n = 0, 1, 2, \dots,$$

which satisfy the recursion

$$(n+1)^3 u_{n+1} - (2n+1)(17n^2 + 17n + 5)u_n + n^3 u_{n-1} = 0.$$

Newest history

In 2011, Zhi-Wei Sun (and Gert Almkvist) experimentally observed several new identities for $1/\pi$ of the form

$$\sum_{n=0}^{\infty} \frac{(s)_n(1-s)_n}{n!^2} (A + Bn) T_n(b, c) \lambda^n = \frac{C}{\pi},$$

where $s \in \{1/2, 1/3, 1/4\}$, $A, B, b, c \in \mathbb{Z}$, $T_n(b, c)$ denotes the coefficient of x^n in the expansion of $(x^2 + bx + c)^n$, viz.

$$T_n(b, c) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k} b^{n-2k} c^k,$$

while λ and C are either rational or (linear combinations of) quadratic irrationalities.

Sun's identities

Examples:

$$\sum_{n=0}^{\infty} \binom{2n}{n}^2 (7 + 30n) \frac{T_n(34, 1)}{(-2^{10})^n} = \frac{12}{\pi},$$

$$\sum_{n=0}^{\infty} \frac{(3n)!}{n!^3} (1 + 18n) \frac{T_n(730, 729)}{30^{3n}} = \frac{25\sqrt{3}}{\pi},$$

$$\sum_{n=0}^{\infty} \frac{(4n)!}{(2n)!n!^2} (13 + 80n) \frac{T_n(7, 4096)}{(-168^2)^n} = \frac{14\sqrt{210} + 21\sqrt{42}}{8\pi},$$

$$\sum_{n=0}^{\infty} \binom{2n}{n}^2 (1 + 10n) \frac{T_{2n}(38, 1)}{240^{2n}} = \frac{15\sqrt{6}}{4\pi},$$

$$\sum_{n=0}^{\infty} \frac{(3n)!}{n!^3} (277 + 1638n) \frac{T_{3n}(62, 1)}{(-240)^{3n}} = \frac{44\sqrt{105}}{\pi}.$$

Legendre polynomials

The binomial sums $T_n(b, c)$ can be expressed via the classical Legendre polynomials

$$P_n(x) = {}_2F_1\left(\begin{matrix} -n, n+1 \\ 1 \end{matrix} \middle| \frac{1-x}{2}\right)$$

by means of the formula

$$T_n(b, c) = (b^2 - 4c)^{n/2} P_n\left(\frac{b}{(b^2 - 4c)^{1/2}}\right),$$

so that the above equalities assume the form

$$\sum_{n=0}^{\infty} \frac{(s)_n (1-s)_n}{n!^2} (A + Bn) P_n(x_0) z_0^n = \frac{C}{\pi}.$$

Here and below the hypergeometric series is defined by

$${}_mF_{m-1}\left(\begin{matrix} a_1, a_2, \dots, a_m \\ b_2, \dots, b_m \end{matrix} \middle| z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_m)_n}{(b_2)_n \cdots (b_m)_n} \frac{z^n}{n!}.$$

Brafman's generating function

The Legendre polynomials can be alternatively given by the generating function

$$(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)z^n,$$

but there are many other generating functions for them. One particular family of examples is due to Fred Brafman (1951).

Theorem A

The following generating series is valid:

$$\sum_{n=0}^{\infty} \frac{(s)_n(1-s)_n}{n!^2} P_n(x)z^n = {}_2F_1\left(s, 1-s \mid \frac{1-\rho-z}{2}\right) \cdot {}_2F_1\left(s, 1-s \mid \frac{1-\rho+z}{2}\right),$$

where $\rho = \rho(x, z) := (1 - 2xz + z^2)^{1/2}$.

Bailey's identity

Theorem A is a consequence of Bailey's identity for a special case of Appell's hypergeometric function of the fourth type,

$$\sum_{m,k=0}^{\infty} \frac{(s)_{m+k}(1-s)_{m+k}}{m!^2 k!^2} (X(1-Y))^m (Y(1-X))^k \\ = {}_2F_1\left(s, 1-s \mid X\right) \cdot {}_2F_1\left(s, 1-s \mid Y\right).$$

In our joint work with Heng Huat Chan and James Wan we demonstrate how any entry on Sun's list can be derived from Brafman's identity and related modular parametrisations of the (arithmetic) ${}_2F_1$ hypergeometric series.

Apéry-like sequences

In another joint work with James Wan we go further and extend Bailey's identity and Brafman's generating function to more general Apéry-like sequences u_0, u_1, u_2, \dots which satisfy the second order recurrence relation

$$(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1} \quad \text{for } n = 0, 1, 2, \dots, \\ u_{-1} = 0, \quad u_0 = 1,$$

for a given data a, b and c .

The hypergeometric term $u_n = (s)_n(1-s)_n/n!^2$ corresponds to a special degenerate case $c = 0$ and $a = 1, b = s(1-s)$ in the recursion.

Note that the generating series $F(X) = \sum_{n=0}^{\infty} u_n X^n$ for a sequence satisfying the recurrence equation is a unique, analytic at the origin solution of the differential equation

$$(\theta^2 - X(a\theta^2 + a\theta + b) + cX^2(\theta + 1)^2)F(X) = 0, \quad \text{where } \theta = \theta_X := X \frac{\partial}{\partial X}.$$

Generalized Bailey's identity

Our first result concerns the generating function of u_n .

Theorem 1

For the solution u_n of the recurrence equation above, define

$$g(X, Y) = \frac{X(1 - aY + cY^2)}{(1 - cXY)^2}.$$

Then in a neighbourhood of $X = Y = 0$,

$$\left\{ \sum_{n=0}^{\infty} u_n X^n \right\} \left\{ \sum_{n=0}^{\infty} u_n Y^n \right\} = \frac{1}{1 - cXY} \sum_{n=0}^{\infty} u_n \sum_{m=0}^n \binom{n}{m}^2 g(X, Y)^m g(Y, X)^{n-m}.$$

Therefore, Bailey's identity corresponds to the particular choice $c = 0$ in Theorem 1.

Generalised Brafman's identity

Following Brafman's derivation of Theorem A we deduce the following generalized generating functions of Legendre polynomials.

Theorem 2

For the solution u_n of the recurrence equation above, the following identity is valid in a neighbourhood of $X = Y = 0$:

$$\begin{aligned} & \sum_{n=0}^{\infty} u_n P_n \left(\frac{(X+Y)(1+cXY) - 2aXY}{(Y-X)(1-cXY)} \right) \left(\frac{Y-X}{1-cXY} \right)^n \\ &= (1-cXY) \left\{ \sum_{n=0}^{\infty} u_n X^n \right\} \left\{ \sum_{n=0}^{\infty} u_n Y^n \right\}. \end{aligned}$$

Special generating functions

Theorem 3

The following identities are valid in a neighbourhood of $X = Y = 0$:

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{n!^2} P_{2n} \left(\frac{(1-X-Y)(X+Y-2XY)}{(Y-X)(1-X-Y+2XY)} \right) \cdot \left(\frac{X-Y}{1-X-Y+2XY} \right)^{2n} \\ &= (1-X-Y+2XY) {}_2F_1 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \middle| 4X(1-X) \right) \cdot {}_2F_1 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \middle| 4Y(1-Y) \right), \\ & \sum_{n=0}^{\infty} \frac{\left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{n!^2} P_{3n} \left(\frac{(X+Y)(1-X-Y+3XY) - 2XY}{(Y-X)\sqrt{p(X,Y)}} \right) \cdot \left(\frac{X-Y}{\sqrt{p(X,Y)}} \right)^{3n} \\ &= \frac{\sqrt{p(X,Y)}}{(1-3X)(1-3Y)} {}_2F_1 \left(\begin{matrix} \frac{1}{3}, \frac{2}{3} \\ 1 \end{matrix} \middle| -\frac{9X(1-3X+3X^2)}{(1-3X)^3} \right) \\ & \quad \times {}_2F_1 \left(\begin{matrix} \frac{1}{3}, \frac{2}{3} \\ 1 \end{matrix} \middle| -\frac{9Y(1-3Y+3Y^2)}{(1-3Y)^3} \right), \end{aligned}$$

where $p(X, Y) = (1 - X - Y + 3XY)^2 - 4XY$.

More observations from Sun

Because Sun's preprint is a dynamic survey of continuous experimental discoveries by its author, a few newer examples for $1/\pi$ involving the Legendre polynomials appeared after acceptance of our papers with Heng Huat Chan and James Wan.

Namely, the two groups of identities (VI1)–(VI3) and (VII1)–(VII7) related to the generating functions

$$\sum_{n=0}^{\infty} P_n(y)^3 z^n \quad \text{and} \quad \sum_{n=0}^{\infty} \binom{2n}{n} P_n(y)^2 z^n$$

are now given on p. 23 of the 37th edition of Sun's preprint.

Some classics and its revision

The closest-to-wanted identity is Bailey's

$$\sum_{n=0}^{\infty} P_n(x)P_n(y)z^n = \frac{1}{(1 + z(z - 2\sqrt{(1-x^2)(1-y^2)} - 2xy))^{1/2}} \\ \times {}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \middle| \frac{-4\sqrt{(1-x^2)(1-y^2)}z}{1 + z(z - 2\sqrt{(1-x^2)(1-y^2)} - 2xy)}\right)$$

or in a nicer form due to James Wan:

$$\sum_{n=0}^{\infty} P_n(x)P_n(y)z^n = \frac{1}{(1 - 2xyz + z^2)^{1/2}} {}_2F_1\left(\begin{matrix} \frac{1}{4}, \frac{3}{4} \\ 1 \end{matrix} \middle| \frac{4(1-x^2)(1-y^2)z^2}{(1 - 2xyz + z^2)^2}\right) \\ = \sum_{n=0}^{\infty} \binom{4n}{2n} \binom{2n}{n} \frac{(1-x^2)^n(1-y^2)^nz^{2n}}{2^{4n}(1 - 2xyz + z^2)^{2n+1/2}}.$$

Unfortunately, no simple generalisation of the result for the terms on the left-hand side twisted by the central binomial coefficients is known, even in the particular case $x = y$.

Squaring Legendre polynomials

With the help of Clausen's identity

$$P_n(y)^2 = {}_3F_2\left(\begin{matrix} -n, n+1, \frac{1}{2} \\ 1, 1 \end{matrix} \middle| 1-y^2\right) = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{n} \binom{2k}{k} \left(-\frac{1-y^2}{4}\right)^k,$$

we find that one of the wanted generating functions is equivalent to

$$\sum_{n=0}^{\infty} \binom{2n}{n} z^n \sum_{k=0}^n \binom{n}{k} \binom{n+k}{n} \binom{2k}{k} x^k.$$

In view of our earlier results, it is likely that the latter generating function can be written as a product of two arithmetic hypergeometric series, each satisfying a second order linear differential equation.

However we can only recover the special case $z = x/(1+x)^2$ of the expected identity, the case which is suggested by Sun's observations (VII1) and (VII3)–(VII6).

Main theorem

Theorem 4

For v from a small neighbourhood of the origin, take

$$x(v) = \frac{v}{1 + 5v + 8v^2} \quad \text{and} \quad z(v) = \frac{x(v)}{(1 + x(v))^2} = \frac{v(1 + 5v + 8v^2)}{(1 + 2v)^2(1 + 4v)^2}.$$

Then

$$\sum_{n=0}^{\infty} \binom{2n}{n} z(v)^n \sum_{k=0}^n \binom{n}{k} \binom{n+k}{n} \binom{2k}{k} x(v)^k = \frac{1 + 2v}{1 + 4v} \sum_{n=0}^{\infty} u_n \left(\frac{v}{(1 + 4v)^3} \right)^n,$$

where

$$u_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{n} \binom{2k}{n} = \sum_{k=0}^n (-1)^{n-k} \binom{3n+1}{n-k} \binom{n+k}{n}^3$$

is sequence A183204 in OEIS.

Modular parametrisation

Note that the sequence u_n satisfies the Apéry-like recurrence equation

$$(n+1)^3 u_{n+1} = (2n+1)(13n^2 + 13n + 4)u_n + 3n(3n-1)(3n+1)u_{n-1}$$

for $n = 0, 1, 2, \dots$, $u_{-1} = 0$, $u_0 = 1$.

Shaun Cooper constructs a modular parametrisation of the generating function $\sum_{n=0}^{\infty} u_n w^n$ by level 7 modular functions. Namely, he proves that the substitution

$$w(\tau) = \frac{\eta(\tau)^4 \eta(7\tau)^4}{\eta(\tau)^8 + 13\eta(\tau)^4 \eta(7\tau)^4 + 49\eta(7\tau)^8}$$

translates the function into the Eisenstein series $(7E_2(7\tau) - E_2(\tau))/6$. Here $\eta(\tau) = q^{1/24} \prod_{m=1}^{\infty} (1 - q^m)$ is Dedekind's eta function, $q = e^{2\pi i \tau}$, and

$$E_2(\tau) = \frac{12}{\pi i} \frac{d \log \eta}{d\tau} = 1 - 24 \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n}.$$

Variations of main theorem

An equivalent form of the identity is

$$\begin{aligned} \sum_{n=0}^{\infty} \binom{2n}{n} P_n \left(\frac{\sqrt{(1+v)(1+8v)}}{\sqrt{1+5v+8v^2}} \right)^2 \left(\frac{v(1+5v+8v^2)}{(1+2v)^2(1+4v)^2} \right)^n \\ = \frac{1+2v}{1+4v} \sum_{n=0}^{\infty} u_n \left(\frac{v}{(1+4v)^3} \right)^n. \end{aligned}$$

Note the hypergeometric expressions of $\sum_{n=0}^{\infty} u_n w^n$ which follow from earlier results of Chan and Cooper:

$$\begin{aligned} \frac{1}{\sqrt{1+13h+49h^2}} \sum_{n=0}^{\infty} u_n \left(\frac{h}{1+13h+49h^2} \right)^n \\ = \frac{1}{\sqrt{1+245h+2401h^2}} {}_3F_2 \left(\begin{matrix} \frac{1}{6}, \frac{1}{2}, \frac{5}{6} \\ 1, 1 \end{matrix} \middle| \frac{1728h}{(1+13h+49h^2)(1+245h+2401h^2)^3} \right) \\ = \frac{1}{\sqrt{1+5h+h^2}} {}_3F_2 \left(\begin{matrix} \frac{1}{6}, \frac{1}{2}, \frac{5}{6} \\ 1, 1 \end{matrix} \middle| \frac{1728h^7}{(1+13h+49h^2)(1+5h+h^2)^3} \right). \end{aligned}$$

Satellite identity

Theorem 5

The identity

$$\sum_{n=0}^{\infty} \binom{2n}{n} \left(\frac{x}{(1+x)^2} \right)^n \sum_{k=0}^n \binom{n}{k} \binom{n+k}{n} \binom{2k}{k} x^k \\ \times (2x(3+4x) - n(1-x)(3+5x) + 4k(1+x)(1+4x)) = 0$$

is valid whenever the left-hand side makes sense.

Using Cooper's parametrisation, Theorems 4 and 5 we derive of Sun's identities (VII1) and (VII3)–(VII6) for $1/\pi$.

How far all this is generalisable

It is apparent that there is a variety of formulae similar to the ones in Theorems 4, 5 and designed for generating functions of other polynomials. For example, Sun's list contains five identities involving values of the polynomials

$$A_n(x) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{n} x^k, \quad n = 0, 1, 2, \dots$$

By examining three of the entries one notices that the parameters x and z of the generating function

$$\sum_{n=0}^{\infty} \binom{2n}{n} A_n(x) z^n$$

are related by $z = x/(1 - 4x)$, while the other entries are subject to the relation $z = 1/(x + 1)^2$. With some work we find that those specialisations indeed lead to third order arithmetic linear differential equations which can be then identified with the known examples.

General question

On the other hand, it is natural to expect the existence of Bailey–Brafman-like identities (Theorems 1–3) for the two-variate generating functions of this type.

Question

Given an (arithmetic) generating function $\sum_{n=0}^{\infty} A_n z^n$ which satisfies a second order linear differential equation (with regular singularities), is it true that $\sum_{n=0}^{\infty} \binom{2n}{n} A_n z^n$ can be written as the product of two arithmetic series, each satisfying (its own) second order linear differential equation?

Here, of course, we allow A_n depend on some other parameters.

The example of such a product decomposition for

$A_n = A_n(x) = \sum_k \binom{n}{k}^2 \binom{2k}{n} x^n$ is given recently by Mat Rogers and Armin Straub.

An affirmative answer to the question will give one an arithmetic parametrisation of the generating function $\sum_{n=0}^{\infty} \binom{2n}{n} P_n(x) P_n(y) z^n \dots$

And much more!

Merci

Thank you!