

Binomial Sums Related to Rational Approximations to $\zeta(4)$

V. V. Zudilin

Received December 16, 2003

KEY WORDS: *binomial sum, rational approximation, polynomial recursion, harmonic numbers, hypergeometric series, zeta function.*

Consider the following 3-term polynomial recursion:

$$(n+1)^5 u_{n+1} - 3(2n+1)(3n^2+3n+1)(15n^2+15n+4)u_n - 3n^3(3n-1)(3n+1)u_{n-1} = 0 \quad \text{for } n \geq 1,$$

and take the two linearly independent solutions $\{u_n\}_{n=0}^\infty$ and $\{v_n\}_{n=0}^\infty$, determined by the initial conditions $u_0 = 1$, $u_1 = 12$ and $v_0 = 0$, $v_1 = 13$. In [1] we give a hypergeometric interpretation of the sequence $u_n \zeta(4) - v_n$, $n = 0, 1, 2, \dots$, from which we obtain the limit

$$\lim_{n \rightarrow \infty} \frac{v_n}{u_n} = \zeta(4) = \frac{\pi^4}{90}$$

and the representation

$$\begin{aligned} u_n &= (-1)^{n+1} \sum_{l=0}^n \frac{d}{dl} \left(\frac{n}{2} - l \right) \binom{n}{l}^4 \binom{n+l}{n}^2 \binom{2n-l}{n}^2 \\ &= (-1)^n \sum_{l=0}^n \left(\frac{n}{2} - l \right) \binom{n}{l}^4 \binom{n+l}{n}^2 \binom{2n-l}{n}^2 \\ &\quad \times \left(\frac{1}{n/2-l} - 6H_{n-l} + 6H_l - 2H_{n+l} + 2H_{2n-l} \right), \end{aligned} \tag{1}$$

where $H_l = \sum_{j=1}^l j^{-1}$ are harmonic numbers. The fact that all the u_n are integers (conjectured in [1]) is not an immediate consequence of formula (1). In the recent work [2], C. Krattenthaler and T. Rivoal prove (among several other useful theorems and beautiful binomial identities) that

$$u_n = \sum_{i,j} \binom{n}{i}^2 \binom{n}{j}^2 \binom{n+j}{n} \binom{n+j-i}{n} \binom{2n-i}{n}, \quad n = 0, 1, 2, \dots,$$

from which one has the desired inclusions $u_n \in \mathbb{Z}$. The main objective of the present note is to give a simpler proof of the formula for the numbers u_n as well as to indicate some other representations that also show that all the u_n are integers.

We use the standard notation

$${}_{r+1}F_r \left(\begin{matrix} a_0, a_1, \dots, a_r \\ b_1, \dots, b_r \end{matrix} \middle| z \right) = \sum_{l=0}^{\infty} \frac{(a_0)_l (a_1)_l \cdots (a_r)_l}{l! (b_1)_l \cdots (b_r)_l} z^l$$

for the generalized hypergeometric series; the notation $(a)_l = a(a+1)\cdots(a+l-1)$ for $l = 1, 2, \dots$ and $(a)_0 = 1$ stands for the Pochhammer symbol.

The following formula is due G. E. Andrews. Passing to the limit as $q \rightarrow 1$ in [3, Theorem 4] (see also [4] for a related application of the identity), we obtain: for $s \geq 1$ and m a nonnegative integer,

$$\begin{aligned} & {}_{2s+3}F_{2s+2} \left(\begin{matrix} a, 1 + \frac{1}{2}a, & b_1, & c_1, & \dots, & b_s, & c_s, & -m \\ \frac{1}{2}a, & 1 + a - b_1, & 1 + a - c_1, & \dots, & 1 + a - b_s, & 1 + a - c_s, & 1 + a + m \end{matrix} \middle| 1 \right) \\ &= \frac{(1+a)_m (1+a-b_s-c_s)_m}{(1+a-b_s)_m (1+a-c_s)_m} \sum_{l_1 \geq 0} \frac{(1+a-b_1-c_1)_{l_1} (b_2)_{l_1} (c_2)_{l_1}}{l_1! (1+a-b_1)_{l_1} (1+a-c_1)_{l_1}} \\ &\quad \times \sum_{l_2 \geq 0} \frac{(1+a-b_2-c_2)_{l_2} (b_3)_{l_1+l_2} (c_3)_{l_1+l_2}}{l_2! (1+a-b_2)_{l_1+l_2} (1+a-c_2)_{l_1+l_2}} \dots \\ &\quad \times \sum_{l_{s-1} \geq 0} \frac{(1+a-b_{s-1}-c_{s-1})_{l_{s-1}} (b_s)_{l_1+\dots+l_{s-1}} (c_s)_{l_1+\dots+l_{s-1}}}{l_{s-1}! (1+a-b_{s-1})_{l_1+\dots+l_{s-1}} (1+a-c_{s-1})_{l_1+\dots+l_{s-1}}} \frac{(-m)_{l_1+\dots+l_{s-1}}}{(b_s+c_s-a-m)_{l_1+\dots+l_{s-1}}}. \end{aligned}$$

Taking $s = 3$, $a = -n - 2\varepsilon$, $b_1 = b_2 = b_3 = c_2 = -n - \varepsilon$, $c_1 = c_3 = n - \varepsilon + 1$, and $m = n$, $i = l_1$, $j = l_1 + l_2$, from Andrews's formula we obtain

$$\begin{aligned} & \sum_{l=0}^n \frac{-\frac{n}{2} - \varepsilon + l}{-\frac{n}{2} - \varepsilon} \cdot \frac{(-n - 2\varepsilon)_l}{(1)_l} \cdot \frac{(-n)_l}{(1 - 2\varepsilon)_l} \cdot \left(\frac{(1 + n - \varepsilon)_l}{(-2n - \varepsilon)_l} \right)^2 \cdot \left(\frac{(-n - \varepsilon)_l}{(1 - \varepsilon)_l} \right)^4 \\ &= \frac{(1 - n - 2\varepsilon)_n (-n)_n}{(1 - \varepsilon)_n (-2n - \varepsilon)_n} \sum_i \frac{(-n)_i (-n - \varepsilon)_i^2}{i! (1 - \varepsilon)_i (-2n - \varepsilon)_i} \sum_j \frac{(1 + n)_{j-i} (-n - \varepsilon)_j (1 + n - \varepsilon)_j (-n)_j}{(j - i)! (1 - \varepsilon)_j^2 j!}. \end{aligned} \tag{2}$$

Using the trivial equality

$$(1 - n - 2\varepsilon)_n = \frac{-\varepsilon}{-\frac{n}{2} - \varepsilon} \cdot (-n - 2\varepsilon)_n,$$

we may rewrite (2) in the form

$$\begin{aligned} \frac{1}{\varepsilon} \sum_{l=0}^n A_l(\varepsilon) &= \frac{1}{\varepsilon} \sum_{l=0}^n \left(\frac{n}{2} + \varepsilon - l \right) \cdot \frac{(-n - 2\varepsilon)_l}{(1)_l} \cdot \frac{(-n)_l}{(1 - 2\varepsilon)_l} \cdot \left(\frac{(1 + n - \varepsilon)_l}{(-2n - \varepsilon)_l} \right)^2 \cdot \left(\frac{(-n - \varepsilon)_l}{(1 - \varepsilon)_l} \right)^4 \\ &= \frac{(-n - 2\varepsilon)_n (-n)_n}{(1 - \varepsilon)_n (-2n - \varepsilon)_n} \sum_i \frac{(-n)_i (-n - \varepsilon)_i^2}{i! (1 - \varepsilon)_i (-2n - \varepsilon)_i} \\ &\quad \times \sum_j \frac{(1 + n)_{j-i} (-n - \varepsilon)_j (1 + n - \varepsilon)_j (-n)_j}{(j - i)! (1 - \varepsilon)_j^2 j!}. \end{aligned} \tag{3}$$

Now, we let ε tend to 0. On the right-hand side of (3), we only need to plug in $\varepsilon = 0$. To proceed with the left-hand side, we first note that $A_l(0) = -A_{n-l}(0)$ for all $l = 0, 1, \dots, n$; hence

$$\lim_{\varepsilon \rightarrow 0} \sum_{l=0}^n A_l(\varepsilon) = \sum_{l=0}^n A_l(0) = 0, \tag{4}$$

and we may apply the l'Hôpital rule:

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \sum_{l=0}^n A_l(\varepsilon) &= \sum_{l=0}^n \frac{\partial A_l(\varepsilon)}{\partial \varepsilon} \Big|_{\varepsilon=0} = \sum_{l=0}^n A_l(0) \cdot \frac{\partial \text{Log } A_l(\varepsilon)}{\partial \varepsilon} \Big|_{\varepsilon=0} \\
&= \sum_{l=0}^n A_l(0) \cdot \left(\frac{1}{\frac{n}{2} - l} - 2 \sum_{j=1}^l \frac{1}{-n+j-1} + 2 \sum_{j=1}^l \frac{1}{j} - 2 \sum_{j=1}^l \frac{1}{n+j} \right. \\
&\quad \left. + 2 \sum_{j=1}^l \frac{1}{-2n+j-1} - 4 \sum_{j=1}^l \frac{1}{-n+j-1} + 4 \sum_{j=1}^l \frac{1}{j} \right) \\
&= \sum_{l=0}^n A_l(0) \cdot \left(\frac{1}{\frac{n}{2} - l} + 6(H_n - H_{n-l}) + 6H_l - 2(H_{n+l} - H_n) - 2(H_{2n} - H_{2n-l}) \right) \\
&= \sum_{l=0}^n A_l(0) \cdot \left(\frac{1}{\frac{n}{2} - l} - 6H_{n-l} + 6H_l - 2H_{n+l} + 2H_{2n-l} \right),
\end{aligned}$$

where, on the last step, we use the following consequence of (4):

$$\sum_{l=0}^n A_l(0)H_n = \sum_{l=0}^n A_l(0)H_{2n} = 0.$$

Since

$$A_l(0) \cdot \left(\frac{1}{\frac{n}{2} - l} - 6H_{n-l} + 6H_l - 2H_{n+l} + 2H_{2n-l} \right) = -\frac{d}{dl} A_l(0),$$

after developing all Pochhammer symbols in the $\varepsilon \rightarrow 0$ form of (3), we arrive at the following identity from [2, Sec. 13]:

$$\begin{aligned}
& - \sum_{l=0}^n \frac{d}{dl} \left(\frac{n}{2} - l \right) \binom{n}{l}^4 \binom{n+l}{n}^2 \binom{2n-l}{n}^2 \\
&= (-1)^n \sum_i \binom{n}{i}^2 \binom{2n-i}{n} \sum_j \binom{n+j-i}{n} \binom{n}{j}^2 \binom{n+j}{n}. \tag{5}
\end{aligned}$$

Clearly, the left-hand side of Andrews's formula is symmetric with respect to the group of parameters $b_1, c_1, b_2, c_2, b_3, c_3$. Therefore, setting as before $a = -n - 2\varepsilon$, $m = n$, and all the parameters of the group to be $-n - \varepsilon$, except the following two:

- (a) $b_1 = c_1 = n - \varepsilon + 1$; (b) $b_2 = c_2 = n - \varepsilon + 1$; (c) $b_3 = c_3 = n - \varepsilon + 1$;
(d) $c_1 = c_2 = n - \varepsilon + 1$; (e) $c_2 = c_3 = n - \varepsilon + 1$; (f) $c_1 = c_3 = n - \varepsilon + 1$

(the last case corresponds to the above identity (5)), we arrive at the five more representations of the left-hand side of (5):

$$\begin{aligned}
(-1)^n u_n &= - \sum_{l=0}^n \frac{d}{dl} \left(\frac{n}{2} - l \right) \binom{n}{l}^4 \binom{n+l}{n}^2 \binom{2n-l}{n}^2 \\
&= (-1)^n \sum_i (-1)^i \binom{3n+1}{i} \binom{2n-i}{n}^2 \sum_j \binom{n+j-i}{n} \binom{n}{j}^2 \binom{2n-j}{n}
\end{aligned}$$

$$\begin{aligned}
&= (-1)^n \sum_i (-1)^i \binom{n+i}{n}^3 \sum_j (-1)^j \binom{3n+1}{j-i} \binom{2n-j}{n}^3 \\
&= \sum_i \binom{n}{i}^2 \binom{n+i}{n} \sum_j (-1)^j \binom{n+j-i}{n} \binom{n+j}{n}^2 \binom{3n+1}{n-j} \\
&= (-1)^n \sum_i \binom{n}{i} \binom{n+i}{n} \binom{2n-i}{n} \sum_j \binom{n}{j-i} \binom{n}{j} \binom{2n-j}{n}^2 \\
&= (-1)^n \sum_i \binom{n}{i} \binom{n+i}{n}^2 \sum_j \binom{n}{j-i} \binom{n}{j} \binom{n+j}{n} \binom{2n-j}{n}.
\end{aligned}$$

ACKNOWLEDGMENTS

The work is supported by an Alexander von Humboldt research fellowship and in part by the Russian Foundation for Basic Research under grant no. 03-01-00359.

REFERENCES

1. W. Zudilin (V. V. Zudilin), *Electron. J. Combin.*, **10** (2003), no. 1, #R14, 10 pp. (electronic).
2. C. Krattenthaler and T. Rivoal, "Hypergéométrie et fonction zêta de Riemann," in: *E-print math.NT/0311114*, 2003.
3. G. E. Andrews, in: *Theory and Application of Special Functions (Proc. Advanced Sem., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1975)*, (R. A. Askey, editor), Math. Res. Center, Univ. Wisconsin, Publ. no. 35, Academic Press, New York, 1975, pp. 191–224.
4. W. Zudilin (V. V. Zudilin), *Electron. J. Combin.*, **11** (2004), no. 1, #R22, 8 pp. (electronic).

M. V. LOMONOSOV MOSCOW STATE UNIVERSITY
E-mail: wadim@ips.ras.ru