Mathematical Notes, vol. 76, no. 2, 2004, pp. 276–279.
Translated from Matematicheskie Zametki, vol. 76, no. 2, 2004, pp. 300–303.

BRIEF COMMUNICATIONS

The Inverse Legendre Transform of a Certain Family of Sequences

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Received April 27, 2004

Key words: *hypergeometric series, binomial sum, Legendre transform, Apéry's numbers.*

The *Legendre transform* of a numerical sequence $\{c_n\}_{n=0}^{\infty}$ is defined by the rule

$$a_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} c_k, \qquad n = 0, 1, 2, \dots ;$$

this procedure transforms the sequence $\{c_n\}_{n=0}^{\infty}$ of integers into another integer-valued sequence $\{a_n\}_{n=0}^{\infty}$. In general, the inverse Legendre transform [1]

$$c_n = \binom{2n}{n}^{-1} \sum_{k=0}^{n} (-1)^{n-k} d_{n,k} a_k, \qquad n = 0, 1, 2, \dots, \tag{1}$$

where

$$d_{n,k} = \binom{2n}{n-k} - \binom{2n}{n-k-1} = \frac{2k+1}{n+k+1}\binom{2n}{n-k}, \qquad n = 0, 1, 2, \dots, \quad k = 0, 1, \dots, n, \tag{2}$$

does not possess the above property, but, in accordance with (1), the inclusions $D_n c_n \in \mathbb{Z}$, $n = 1, 2, \dots$, follow from $\{a_n\}_{n=0}^{\infty} \subset \mathbb{Z}$. Here the symbol D_n stands for the least common multiple of the numbers $1, 2, \dots, n$. On the basis of experimental data, A. Schmidt conjectured that, for a certain family of sequences $\{c_n\}_{n=1}^{\infty}$, the inverse Legendre transform results in integer-valued sequences. Namely, he formulated the following problem in [2, Sec. 4].

Schmidt's Problem. For any integer $r \ge 2$, define a sequence of numbers $\{c_k^{(r)}\}_{k=0,1,\dots}$, independent of the parameter n, by the relation

$$\sum_{k=0}^{n} \binom{n}{k}^r \binom{n+k}{k}^r = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} c_k^{(r)}, \qquad n = 0, 1, 2, \dots .$$

Show that all the numbers $c_k^{(r)}$ are integers.

Note that the sequence $\{c_k^{(2)}\}$ in the case $r = 2$ gives the inverse Legendre transform of the famous sequence of Apéry's numbers [3] expressing the denominators of the convergents in his proof of the irrationality of the number $\zeta(3) = \sum_{n=1}^{\infty} n^{-3}$.

 0001-4346/2004/7612-0276

The problem was almost immediately solved after its communication by Schmidt himself [1] in the case $r=2$ and by Strehl [4] when $r=2,3$. Namely, the following explicit expressions demonstrating that the corresponding sequences are integer were obtained:

$$c_k^{(2)}=\sum_{j=0}^{k}\binom{k}{j}^3=\sum_{j=0}^{k}\binom{k}{j}^2\binom{2j}{k}, \quad c_k^{(3)}=\sum_{j=0}^{k}\binom{2j}{j}^2\binom{k}{j}^2\binom{2j}{k-j}, \qquad k=0,1,2,\dots .$$

(The binomial coefficients $\binom{n}{k}$ are regarded as zero if $k<0$ and $k>n$.) Schmidt's problem was later formulated in [5, Exercise 114 on p. 256] with an indication that H. Wilf had shown the inclusions $c_n^{(r)}\in\mathbb{Z}$ for all r, but only for any n not greater than 9.

The present article is aimed at the complete solution of Schmidt's problem; we prove the following explicit formulas for $c_n^{(r)}$.

Theorem. *The numbers $c_n^{(r)}$ in the statement of Schmidt's problem are indeed, integers. In particular, for any $s=1,2,\dots$ we have the formulas*

$$\begin{aligned} c_n^{(2s)}&=\sum_{j=0}^{n}\binom{2j}{j}^{2s-1}\binom{n}{j}\sum_{k_1\ge\dots\ge k_{s-1}\ge j}\cdots\sum\binom{j}{n-k_1}\binom{k_1}{j}\binom{k_1+j}{k_1-j}\\ &\quad\times\binom{2j}{k_{s-1}-j}\prod_{i=2}^{s-1}\binom{2j}{k_{i-1}-k_i}\binom{k_i+j}{k_i-j}^2,\\ c_n^{(2s+1)}&=\sum_{j=0}^{n}\binom{2j}{j}^{2s}\binom{n}{j}^2\sum_{k_1\ge\dots\ge k_{s-1}\ge j}\cdots\sum\binom{2j}{n-k_1}\binom{k_1+j}{k_1-j}^2\\ &\quad\times\binom{2j}{k_{s-1}-j}\prod_{i=2}^{s-1}\binom{2j}{k_{i-1}-k_i}\binom{k_i+j}{k_i-j}^2, \end{aligned}$$

where $n=0,1,2,\dots$.

Proof. Applying the inverse Legendre transform (1), (2) to the sequence

$$\sum_{k=0}^{n}\binom{n}{k}^r\binom{n+k}{k}^r=\sum_{k=0}^{n}\binom{2k}{k}^r\binom{n+k}{n-k}^r, \qquad n=0,1,2,\dots,$$

we deduce

$$c_n^{(r)}=\sum_{j=0}^{n}\binom{2n}{n}^{-1}\binom{2j}{j}^r t_{n,j}^{(r)}, \qquad n=0,1,2,\dots, \tag{3}$$

where

$$\begin{aligned} t_{n,j}^{(r)}&=\sum_{k=j}^{n}(-1)^{n-k}d_{n,k}\binom{k+j}{k-j}^r\\ &=\sum_{l\ge0}(-1)^l\frac{2n-2l+1}{2n-l+1}\binom{2n}{l}\binom{n-l+j}{n-l-j}^r, \qquad n=0,1,2,\dots, \quad j=0,1,\dots,n. \end{aligned} \tag{4}$$

The latter sum is a particular case of a *terminating* (i.e., containing a finite number of terms) *hypergeometric series*

$${}_{r+1}F_r\left(\begin{matrix}a_0,a_1,\dots,a_r\\ b_1,\dots,b_r\end{matrix}\,\middle|\,z\right)=\sum_{l=0}^{\infty}\frac{(a_0)_l(a_1)_l\cdots(a_r)_l}{l!\,(b_1)_l\cdots(b_r)_l}z^l$$

for a suitable choice of its parameters; here and below, $(a)_l$ denotes the Pochhammer symbol: $(a)_l=a(a+1)\cdots(a+l-1)$ for $l=1,2,\dots$ and $(a)_0=1$. The ratio of two consecutive terms of the sum in (4) is equal to

$$\frac{-(2n+1)+l}{1+l}\cdot\frac{-\frac12(2n-1)+l}{-\frac12(2n+1)+l}\cdot\biggl(\frac{-(n-j)+l}{-(n+j)+l}\biggr)^r;$$

hence

$$t_{n,j}^{(r)}=\binom{n+j}{n-j}^r\cdot{}_{r+1}F_r\biggl(\begin{matrix}-(2n+1),-\frac12(2n-1),-(n-j),\dots,-(n-j)\\ -\frac12(2n+1),-(n+j),\dots,-(n+j)\end{matrix}\biggm|1\biggr)$$

is a *very well-poised* hypergeometric series. Therefore, our strategy for proving the theorem is as follows: we apply an appropriate transformation formula to the obtained hypergeometric representation of the quantities (4). The required formula, a multiple generalization of the classical Whipple transformation [6], was established by Andrews in [7]. Making the passage $q\to1$ in [7, Theorem 4], we find that *for $s\ge1$ and m a non-negative integer*

$$\begin{aligned}&{}_{2s+3}F_{2s+2}\biggl(\begin{matrix}a,1+\frac12a, & b_1, & c_1, & \dots, & b_s, & c_s, & -m\\ \frac12a, & 1+a-b_1, & 1+a-c_1, & \dots, & 1+a-b_s, & 1+a-c_s, & 1+a+m\end{matrix}\biggm|1\biggr)\\
&\quad=\frac{(1+a)_m(1+a-b_s-c_s)_m}{(1+a-b_s)_m(1+a-c_s)_m}\sum_{l_1\ge0}\sum_{l_2\ge0}\cdots\sum_{l_{s-1}\ge0}\frac{(-m)_{l_1+\dots+l_{s-1}}}{(b_s+c_s-a-m)_{l_1+\dots+l_{s-1}}}\\
&\quad\qquad\times\prod_{i=1}^{s-1}\frac{(1+a-b_i-c_i)_{l_i}(b_{i+1})_{l_1+\dots+l_i}(c_{i+1})_{l_1+\dots+l_i}}{l_i!\,(1+a-b_i)_{l_1+\dots+l_i}(1+a-c_i)_{l_1+\dots+l_i}}.\end{aligned}\tag{5}$$

If $r=2s+1$, then we set $a=-(2n+1)$ and $b_1=c_1=\dots=b_s=c_s=-m=-(n-j)$ in (5) to obtain

$$\begin{aligned}t_{n,j}^{(2s+1)}&=\binom{n+j}{n-j}^{2s-2}\frac{(2n)!}{(3j-n)!\,(n-j)!^3}\sum_{l_1,\dots,l_{s-1}\ge0}\cdots\sum\frac{(-1)^{l_1+\dots+l_{s-1}}(-(n-j))_{l_1+\dots+l_{s-1}}}{(3j-n+1)_{l_1+\dots+l_{s-1}}}\\
&\qquad\times\prod_{i=1}^{s-1}\binom{2j}{l_i}\biggl(\frac{(-(n-j))_{l_1+\dots+l_i}}{(-(n+j))_{l_1+\dots+l_i}}\biggr)^2\\
&=\frac{(2n)!}{(2j)!\,(n-j)!^2}\sum_{l_1,\dots,l_{s-1}\ge0}\cdots\sum\binom{2j}{n-l_1-\dots-l_{s-1}-j}\prod_{i=1}^{s-1}\binom{2j}{l_i}\binom{n-l_1-\dots-l_i+j}{n-l_1-\dots-l_i-j}^2.\end{aligned}$$

If $r=2s$, apply formula (5) taking $a=-(2n+1)$, $b_1=(a+1)/2=-n$ and $c_1=b_2=\dots=b_s=c_s=-m=-(n-j)$:

$$\begin{aligned}t_{n,j}^{(2s)}&=\binom{n+j}{n-j}^{2s-3}\frac{(2n)!}{(3j-n)!\,(n-j)!^3}\sum_{l_1,\dots,l_{s-1}\ge0}\cdots\sum\frac{(-1)^{l_1+\dots+l_{s-1}}(-(n-j))_{l_1+\dots+l_{s-1}}}{(3j-n+1)_{l_1+\dots+l_{s-1}}}\\
&\qquad\times\binom{j}{l_1}\frac{(-(n-j))_{l_1}}{(-n)_{l_1}}\,\frac{(-(n-j))_{l_1}}{(-(n+j))_{l_1}}\prod_{i=2}^{s-1}\binom{2j}{l_i}\biggl(\frac{(-(n-j))_{l_1+\dots+l_i}}{(-(n+j))_{l_1+\dots+l_i}}\biggr)^2\\
&=\frac{(2n)!\,j!}{n!\,(n-j)!\,(2j)!}\sum_{l_1,\dots,l_{s-1}\ge0}\cdots\sum\binom{2j}{n-l_1-\dots-l_{s-1}-j}\\
&\qquad\times\binom{j}{l_1}\binom{n-l_1}{j}\binom{n-l_1+j}{n-l_1-j}\prod_{i=2}^{s-1}\binom{2j}{l_i}\binom{n-l_1-\dots-l_i+j}{n-l_1-\dots-l_i-j}^2.\end{aligned}$$

Substituting the resulted expressions into (3) completes the proof of the theorem. □

ACKNOWLEDGMENTS

The author thanks P. Bundschuh, C. Krattenthaler, J. Sondow and A. Schmidt for stimulating discussions. Special gratitude is due to the Editorial Board of *The Electronic Journal of Combinatorics* (http://www.combinatorics.org) for the electronic publication in [8] of the result.

This work was supported by an Alexander von Humboldt research fellowship and supported in part by the Russian Foundation for Basic Research under grant no. 03-01-00359.

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