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Derivatives of Siegel modular forms and exponential functions

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Abstract. We show that the differential field generated by Siegel modular forms and the differential field generated by exponentials of polynomials are linearly disjoint over $\mathbb{C}$. Combined with our previous work [3], this provides a complete multidimensional extension of Mahler's theorem on the transcendence degree of the field generated by the exponential function and the derivatives of a modular function. We give two proofs of our result, one purely algebraic, the other analytic, but both based on arguments from differential algebra and on the stability under the action of the symplectic group of the differential field generated by rational and modular functions.

§ 1. Introduction and statement of results

In 1969, Mahler [1] proved that, for any non-constant modular function $f\colon \{\tau\in\mathbb{C} : \Im\tau>0\}\to\mathbb{C}$ and any non-zero complex number c, the five functions

$$\tau,\ q(\tau)=e^{c\tau},\ f(\tau),\ f'(\tau),\ \text{and}\ f''(\tau)$$

are algebraically independent over $\mathbb{C}$. Here the prime denotes differentiation with respect to τ. Since $f'''(\tau)$ is rational over $\mathbb{C}(f(\tau),f'(\tau),f''(\tau))$, it follows that each of the fields

$$\mathbb{C}(f(\tau),f'(\tau),f''(\tau)),\quad \mathbb{C}(\tau,f(\tau),f'(\tau),f''(\tau)),$$
$$\mathbb{C}(q(\tau),f(\tau),f'(\tau),f''(\tau)),\quad \mathbb{C}(\tau,q(\tau),f(\tau),f'(\tau),f''(\tau))$$

is differentially stable, and their transcendence degrees over $\mathbb{C}$ are respectively equal to $3,4,4,5$. Mahler's result was extended on the one hand to more general automorphic functions in one variable by Nishioka [2], and on the other to Siegel modular functions of arbitrary degree in our paper [3]. In the latter case, however, the corresponding collection did not contain the exponential function. The aim of the present paper is to complete our generalization [3] by adding exponentials.

To state our results explicitly, we use the same notation as in [3]. Let g be a positive integer (called the *degree* or *genus*), let k be an algebraically closed subfield of $\mathbb{C}$, and make the following definitions.

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$\mathfrak{H}_g$ = the Siegel half-space of degree g. The $\mathbb{Q}$-vector group Z_g formed by the symmetric matrices of order g has dimension

$$n := \frac{g(g+1)}{2},$$

and $\mathfrak{H}_g$ is open in $Z_g(\mathbb{C})$.

$\boldsymbol{\tau} = (\tau_{jl})_{1\leqslant j\leqslant l\leqslant g}$ is a generic point in $\mathfrak{H}_g$, so that $k(2\pi i\boldsymbol{\tau})$ can be viewed as the field of rational functions on Z_g/k.

$\boldsymbol{\delta} = \{\delta_{jl},\ 1\leqslant j\leqslant l\leqslant g\}$, where

$$\delta_{jl} = \frac{1}{2\pi i}\frac{\partial}{\partial\tau_{jl}}, \quad 1\leqslant j<l\leqslant g, \qquad \text{and} \qquad \delta_{jj} = \frac{1}{\pi i}\frac{\partial}{\partial\tau_{jj}}, \quad 1\leqslant j\leqslant g.$$

These n partial derivatives form a $k(2\pi i\boldsymbol{\tau})$-basis of $Der(k(2\pi i\boldsymbol{\tau})/k)$.

Γ = a congruence subgroup of the symplectic group $Sp_{2g}(\mathbb{Z})$ (equivalently, a subgroup of finite index if $g>1$).

$K := K(\Gamma,k)$ = the field of modular functions with respect to Γ. It is well known that K is a finitely generated extension of k of transcendence degree

$$\operatorname{tr}\deg_k K = \frac{g(g+1)}{2} = n.$$

If $g>1$, the field $K\otimes_k\mathbb{C}$ is identified with the field of meromorphic functions on $\mathfrak{H}_g$ that are invariant under the action of Γ.

$M := M(\Gamma,k)$ = the $\boldsymbol{\delta}$-differential field generated by K, that is, the field generated over k by the partial δ_{jl}-derivatives of all orders of all elements of K.

We proved in Theorem 1 of [3] that the $\boldsymbol{\delta}$-differential field M is a finite extension of the field generated over K by the $\boldsymbol{\delta}$-partial derivatives of order $\leqslant 2$ of the elements of K, and that it has *finite transcendence degree over* k, equal to

$$\operatorname{tr}\deg_k M = \dim Sp_{2g} = 2g^2+g. \tag{1}$$

Furthermore, *M and $\mathbb{C}(\boldsymbol{\tau})$ are linearly disjoint over k*, so that

$$\operatorname{tr}\deg_k M(\boldsymbol{\tau}) = \dim Sp_{2g} + n = \frac{1}{2}g(5g+3). \tag{2}$$

We extend this theorem as follows.

Theorem 1. *Let M be the $\boldsymbol{\delta}$-differential field generated by the field K of modular functions, and let c be an arbitrary non-zero complex number. Then the exponentials $e^{c\tau_{jl}}$, $1\leqslant j\leqslant l\leqslant g$, are algebraically independent over $M(\boldsymbol{\tau})$, whence*

$$\operatorname{tr}\deg_k M(\boldsymbol{\tau}, e^{c\boldsymbol{\tau}}) = \dim Sp_{2g} + 2n = 3g^2+2g.$$

This paper is organized as follows. § 2 contains some preliminaries on the action of Γ on $\mathfrak{H}_g$ and on Z_g. In §§ 3, 4 we give two independent proofs of Theorem 1. More precisely, the proof in § 3 is based on a theorem of Ax (a functional version of Schanuel's conjecture) and is of a purely algebraic nature, while the proof in § 4 uses an easier version of Ax's theorem (namely, Kolchin's multiplicative analogue of Ostrowski's theorem) and an analytic argument. The second proof yields the following sharpening of Theorem 1.

Theorem 2. *In the notation of Theorem* 1, *let* $\{Q_1,\dots,Q_N\}$ *be an arbitrary set of polynomials in* $\boldsymbol{\tau}=(\tau_{jl})_{1\leqslant j\leqslant l\leqslant g}$ *with complex coefficients and no constant terms. Assume that* $Q_1,\dots,Q_N$ *are linearly independent over* $\mathbb{Q}$. *Then the functions* $e^{Q_1(\boldsymbol{\tau})},\dots,e^{Q_N(\boldsymbol{\tau})}$ *are algebraically independent over* $M(\boldsymbol{\tau})$.

Finally, § 5 is devoted to the study of modular thetanulls with special emphasis on the case of genus 2. In this case we explicitly describe the Fourier expansions of some logarithmic derivatives of thetanulls (that is, expansions in terms of the exponential functions $e^{c\tau_{jl}}$ with $c=\pi i$).

Remark 1. Although inspired by the proof in [2], the algebraic and analytic methods of adding exponentials to the differential field generated by modular forms are new, even in the case of genus 1. The second author used similar arguments in another paper [4], where the exponential function is added to the differential field generated by the so-called Yukawa (quantum) coupling.

§ 2. Preliminaries

As mentioned in [3], it is enough to prove our theorems when

$$\Gamma=Sp_{2g}(\mathbb{Z})$$

is the full modular group. Similarly, there is no loss of generality in taking the complex numbers

$$k=\mathbb{C}$$

as the field of constants. These hypotheses will be assumed henceforth.

As in [1] and [2], the following remark plays a crucial role in our proofs. Apart from the finiteness of the transcendence degree of M over $\mathbb{C}$ (see § 1, formula (1) and § 4) or, alternatively, the algebraic independence of τ_{jl} over M (see § 1, formula (2) and § 3), there is only one property of the differential field $M(\boldsymbol{\tau})$ we actually need.[1] We recall that the group Γ has a rational action (on the left) on the Siegel half-space $\mathfrak{H}_g$:

$$(\gamma,\boldsymbol{\tau})\mapsto\gamma\cdot\boldsymbol{\tau}:=(a\boldsymbol{\tau}+b)(c\boldsymbol{\tau}+d)^{-1}\qquad\text{for all}\quad\Bigl(\gamma=\bigl(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\bigr),\boldsymbol{\tau}\Bigr)\in\Gamma\times\mathfrak{H}_g,$$

so that it acts (on the right) on the field $\mathfrak{M}$ of meromorphic functions on $\mathfrak{H}_g$ by

$$(\gamma,f)\mapsto\gamma\cdot f\colon\boldsymbol{\tau}\mapsto(\gamma\cdot f)(\boldsymbol{\tau}):=f(\gamma\cdot\boldsymbol{\tau})\qquad\text{for all}\quad(\gamma,f)\in\Gamma\times\mathfrak{M}.$$

The subfield M of $\mathfrak{M}$ is *not* stable under Γ, but we have the following lemma.

Lemma 1. *The field* $M(\boldsymbol{\tau})$ *is stable under the action of* Γ.

Proof. As in [3], § 4, for each $m=0,1,\dots,\infty$ we denote by $K^{(m)}$ the field generated over $\mathbb{C}$ by the $\boldsymbol{\delta}$-derivatives of order $\leqslant m$ of all the elements of K. Thus, $K^{(0)}=K\subset K^{(1)}\subset\cdots\subset K^{(\infty)}=M$. We shall prove by induction on m that $K^{(m)}(\boldsymbol{\tau})$ is stable under Γ. The definition of K makes this clear for $m=0$ (and for $m=1$,

[1] See Remarks 2, 3 and 4 at the end of § 4 for a more specific comparison of the tools used in our two proofs.

according to [3], § 5, formula (4)). Given any m, let f be an element of $K^{(m)}(\boldsymbol{\tau})$ and let $\gamma \in \Gamma$. We recall that the differential at a point $\boldsymbol{\tau}$ of the automorphism $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of $\mathfrak{H}_g$ is given by

$$\mathrm{d}_{\boldsymbol{\tau}}\gamma = {}^{t}(c\boldsymbol{\tau} + d)^{-1}\mathrm{d}\boldsymbol{\tau}(c\boldsymbol{\tau} + d)^{-1}.$$

Since

$$\mathrm{d}_{\boldsymbol{\tau}}(\gamma \cdot f) = \mathrm{d}_{\gamma\cdot\boldsymbol{\tau}} f \circ \mathrm{d}_{\boldsymbol{\tau}}\gamma, \qquad (\mathrm{d}_{\boldsymbol{\tau}}\gamma)^{-1} = \mathrm{d}_{\gamma\cdot\boldsymbol{\tau}}(\gamma^{-1})$$

and $g = \gamma \cdot f$ lies in $K^{(m)}(\boldsymbol{\tau})$ by the induction hypothesis, we conclude that each of the components $\gamma \cdot (\delta_{jl} f)$ of $\mathrm{d}_{\gamma\cdot\boldsymbol{\tau}} f$ lies in the field generated over $K^{(m)}(\boldsymbol{\tau})$ by the $\boldsymbol{\delta}$-derivatives of order 1 of all elements g of $K^{(m)}(\boldsymbol{\tau})$. But the latter field coincides with $K^{(m+1)}(\boldsymbol{\tau})$, which must therefore be stable under the action of γ.

The remaining assertions in this section are exercises in commutative algebra.

Lemma 2. *The polynomial* $\det \boldsymbol{\tau} \in \mathbb{C}[\tau_{jl},\ 1 \leqslant j \leqslant l \leqslant g]$ *is irreducible.*

Proof (see [5], § 30, Exercise 3 for the corresponding assertion on arbitrary matrices). It is sufficient to prove the irreducibility of the polynomial

$$P_g(\tau_1, \dots, \tau_g) = \det \boldsymbol{\tau}^*, \qquad \text{where} \quad \tau^*_{jl} = \begin{cases} \tau_j & \text{if } j = l, \\ 1 & \text{if } j = l + 1 \text{ or } l = j + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, such a specialization does not decrease the total degree g of $\det \boldsymbol{\tau}$. We prove our claim by induction on g, starting with the polynomials $P_1(\tau_1) = \tau_1$ and $P_2(\tau_1, \tau_2) = \tau_1\tau_2 - 1$, which are clearly irreducible. Expanding $\det \boldsymbol{\tau}^*$ by the last column (or row), we obtain that

$$P_g(\tau_1, \dots, \tau_g) = \tau_g P_{g-1}(\tau_1, \dots, \tau_{g-1}) - P_{g-2}(\tau_1, \dots, \tau_{g-2}).$$

Assume that P_g is reducible. Applying Gauss' lemma to the ring $\mathbb{C}[\tau_1, \dots, \tau_{g-1}][\tau_g]$, we deduce from the induction hypothesis that P_{g-1} and P_{g-2} divide each other. This is clearly impossible, and the proof is complete.

Corollary. *The polynomial* $\det(\boldsymbol{\tau} - m\mathbf{1}_g) \in \mathbb{C}[\boldsymbol{\tau}]$ *is irreducible for any* $m \in \mathbb{Z}$. *In particular, the polynomials* $\det(\boldsymbol{\tau} - m_1\mathbf{1}_g)$ *and* $\det(\boldsymbol{\tau} - m_2\mathbf{1}_g)$ *are coprime in* $\mathbb{C}[\boldsymbol{\tau}]$ *for any pair* m_1, m_2 *of distinct integers.*

Proof. Setting $\boldsymbol{\tau}' = \boldsymbol{\tau} - m\mathbf{1}_g$, we reduce the first assertion to Lemma 2. Then the second assertion follows from the non-vanishing of $\det(\boldsymbol{\tau} - m_2\mathbf{1}_g)$ at $\boldsymbol{\tau} = m_1\mathbf{1}_g$.

In the next two assertions, we consider the left action of Γ on the full vector space Z_g of symmetric matrices (extending the previous action on $\mathfrak{H}_g$) and the corresponding right action of Γ on the field $\mathbb{C}(\boldsymbol{\tau})$ of rational functions on $Z_g/\mathbb{C}$. We shall in particular be interested in elements of Γ of the form

$$\gamma_m = \begin{pmatrix} \mathbf{0}_g & -\mathbf{1}_g \\ \mathbf{1}_g & -m\mathbf{1}_g \end{pmatrix} \in Sp_{2g}(\mathbb{Z}), \qquad \gamma_m(\boldsymbol{\tau}) := \gamma_m \cdot \tau = -(\boldsymbol{\tau} - m\mathbf{1}_g)^{-1},$$

where m denotes any rational integer. For such m we set

$$\mathcal{D}_m = \{\boldsymbol{\tau} \in Z_g : \det(\boldsymbol{\tau} - m\mathbf{1}_g) = 0\}.$$

By the corollary of Lemma 2, this is an irreducible divisor of the affine space Z_g, and the divisors $\mathcal{D}_{m_1}$, $\mathcal{D}_{m_2}$ are distinct for $m_1 \neq m_2$.

Lemma 3. *Let $P \in \mathbb{C}[\boldsymbol{\tau}]$ be a non-zero homogeneous polynomial and let $m \in \mathbb{Z}$. Set $R = \gamma_m \cdot P$, that is, $R(\boldsymbol{\tau}) = P\big(-(\boldsymbol{\tau} - m\mathbf{1}_g)^{-1}\big)$. Then R is a non-zero rational function on Z_g whose polar divisor coincides with* $(\deg P) \cdot \mathcal{D}_m$.

Proof. We may assume by translation that $m = 0$, so that

$$\gamma_0(\boldsymbol{\tau}) = -\boldsymbol{\tau}^{-1} = \left(\frac{T_{jl}}{\det \boldsymbol{\tau}}\right)_{1 \leqslant j,l \leqslant g},$$

where $(-1)^{j+l+1} T_{jl}$ denotes the determinant of the (j,l)-minor of $\boldsymbol{\tau}$. In particular, $T_{jl} = T_{lj}$ are homogeneous polynomials of degree $g-1$ in the ring $\mathbb{C}[\boldsymbol{\tau}]$. Then

$$R(\boldsymbol{\tau}) = P(-\boldsymbol{\tau}^{-1}) = P\left(\frac{T_{jl}}{\det \boldsymbol{\tau}}\right) = \frac{N(\boldsymbol{\tau})}{(\det \boldsymbol{\tau})^{\deg P}},$$

where $N(\boldsymbol{\tau}) = P(T_{jl})$ is either 0 or a homogeneous polynomial in $\mathbb{C}[\boldsymbol{\tau}]$ of degree $(g-1)\deg P$. Since $P \neq 0$ by hypothesis, it is clear that R (and hence N) does not vanish. Consequently, N is a homogeneous polynomial of degree $(g-1)\deg P < \deg(\det \boldsymbol{\tau})^{\deg P}$. Since the divisor $\mathcal{D}_0 = \{\boldsymbol{\tau} \in Z_g : \det \boldsymbol{\tau} = 0\}$ is irreducible by Lemma 2, we conclude that the polar divisor of R in Z_g is a positive integral multiple $\nu \mathcal{D}_0$ of $\mathcal{D}_0$. In other words, $R = \tilde{N}/\tilde{D}$, where $\tilde{D} = (\det \boldsymbol{\tau})^{\nu}$ for some integer $\nu \geqslant 1$ and $\tilde{N}$ is prime to $\tilde{D}$ in $\mathbb{C}[\boldsymbol{\tau}]$. Moreover, $\nu \leqslant \deg P$ and $\deg \tilde{N} < \deg \tilde{D} = g\nu$. This completes the proof in the case when $\deg P = 1$ and reduces the general case to checking that $\nu = \deg P$. Note that the case $\deg P = 1$ already implies that the inverse image $\gamma_0^{-1}(\mathcal{H})$ of a generic affine hyperplane $\mathcal{H} \subset Z_g$ under the automorphism γ_0 of Z_g is a hypersurface of degree g in Z_g. Therefore, the direct image $\gamma_0(\mathcal{L})$ of a generic affine line $\mathcal{L}$ in Z_g is a curve of degree g.

Since $\deg \tilde{N} < \deg \tilde{D}$, the degree $g\nu$ of the polar divisor $\nu \mathcal{D}_0$ of R can be computed as follows. Let α be a generic point in $\mathbb{C}$ and let $\mathcal{L}$ be a generic line in the affine space Z_g. Then the equation $R(\boldsymbol{\tau}) = \alpha$ in Z_g defines an irreducible and reduced divisor D_α of degree $g\nu$ (with the equation $\tilde{N} - \alpha \tilde{D} = 0$), and D_α intersects the line $\mathcal{L}$ properly in $g\nu$ points. Since γ_0 is a local and global isomorphism on Z_g, $\gamma_0(D_\alpha)$ and $\gamma_0(\mathcal{L})$ meet properly at their intersection points: $\gamma_0(D_\alpha) \cap \gamma_0(\mathcal{L}) = \gamma_0(D_\alpha \cap \mathcal{L})$, and there are $g\nu$ such points. But since $R = \gamma_0 \cdot P$, it follows that

$$\gamma_0(D_\alpha) = \{\boldsymbol{\tau}' = \gamma_0(\boldsymbol{\tau}) : R(\boldsymbol{\tau}) = \alpha\} = \{\boldsymbol{\tau}' \in Z_g : P(\boldsymbol{\tau}') = \alpha\}$$

is a divisor of degree $\deg P$ while $\gamma_0(\mathcal{L})$ is a curve of degree g. Furthermore, they intersect at a finite distance (by genericity) and properly (as we have seen). We therefore deduce from Bezout's theorem that $\gamma_0(D_\alpha) \cap \gamma_0(\mathcal{L})$ is a set of $g \deg P$ distinct points. Hence $g\nu = g \deg P$ and $\nu = \deg P$, as required.

Corollary. *Let t be a positive integer, let $P_0, \ldots, P_t \in \mathbb{C}[\boldsymbol{\tau}]$ be (not necessarily homogeneous) non-zero polynomials with no constant terms, and let Q be any non-zero polynomial in $\mathbb{C}[\boldsymbol{\tau}]$. We put $R_m(\boldsymbol{\tau}) = P_m\big(-(\boldsymbol{\tau} - m\mathbf{1}_g)^{-1}\big)$ for each $m = 0, \ldots, t$. Then the rational functions $R_0, \ldots, R_t$ and Q are linearly independent over $\mathbb{C}$.*

Proof. We first remark that any family of non-zero rational functions on Z_g with pairwise distinct polar divisors is linearly independent over $\mathbb{C}$. Here “pairwise distinct" means "either set-theoretically distinct or appearing with distinct multiplicities", and the trivial divisor 0 (that is, the empty set) may appear as one of the divisors. This remark obviously follows from uniqueness of factorization in $\mathbb{C}[\boldsymbol{\tau}]$. Considering each non-zero homogeneous part P_{mi} of degree i (with $i \in I_m \subset \{1,\dots,\deg P_m\}$) of each polynomial P_m $(0 \leqslant m \leqslant t)$ and using Lemma 3 and the corollary of Lemma 2, we get a family of non-zero rational functions $P_{mi}\bigl(-(\boldsymbol{\tau}-m\mathbf{1}_g)^{-1}\bigr)$ which, together with $Q(\tau)$, admit pairwise distinct polar divisors: $i\mathcal{D}_m$ $(m=0,\dots,t,\ i\in I_m)$ and 0. Therefore, this family is linearly independent over $\mathbb{C}$, and so are $R_1,\dots,R_t$, Q.

§ 3. Algebraic proof of Theorem 1

The main tool in this proof is the following result of Ax ([6], Theorem 4), which is a strong functional version of Schanuel's conjecture.

Proposition 1 (Ax's theorem). *Let $F \supseteq E \supseteq k \supseteq \mathbb{Q}$ be a tower of fields and let Δ be a set of derivations of F such that for all $\delta \in \Delta$ we have $\delta E \subseteq E$ and $\bigcap_{\delta\in\Delta}\ker\delta = k$. Let $y_1,\dots,y_s,z_1,\dots,z_s \in F^*$ be elements such that*

(a) *$\delta z_r/z_r - \delta y_r \in E$ for all $\delta \in \Delta$, $r=1,\dots,s$, and*

(b) *no non-trivial product of powers of $z_1,\dots,z_s$ is algebraic over E.*

Then

$$\operatorname{tr}\deg_E E(y_1,\dots,y_s,z_1,\dots,z_s) \geqslant s.$$

Assume that the functions $e^{c\tau_{jl}}$, $1\leqslant j\leqslant l\leqslant g$, are algebraically dependent over $M(\boldsymbol{\tau})$. By Lemma 1, this assumption yields that if $\gamma \in \Gamma$ is any automorphism and we put $(\tau'_{jl}) = \gamma(\tau_{jl})$, then the functions $e^{c\tau'_{jl}}$, $1\leqslant j\leqslant l\leqslant g$, are also algebraically dependent over $M(\boldsymbol{\tau})$. Thus, putting $\boldsymbol{\tau}^{(m)} = (\tau_{jl}^{(m)}) = \gamma_m(\boldsymbol{\tau})$, $m = 0,1,\dots,t$ for arbitrary $\gamma_0,\dots,\gamma_t \in \Gamma$, we obtain the upper estimate

$$\begin{aligned}\operatorname{tr}\deg_{M(\boldsymbol{\tau})} M(\boldsymbol{\tau}, e^{c\boldsymbol{\tau}^{(0)}}, e^{c\boldsymbol{\tau}^{(1)}},\dots,e^{c\boldsymbol{\tau}^{(t)}}) &\leqslant \#\{\boldsymbol{\tau}^{(m)}\}_{m=0,1,\dots,t} - (t+1)\\ &= (t+1)n-(t+1).\end{aligned}$$

We now apply Proposition 1 with

$$k=\mathbb{C},\quad E=M,\quad F=M(\boldsymbol{\tau}, e^{c\boldsymbol{\tau}^{(0)}}, e^{c\boldsymbol{\tau}^{(1)}},\dots,e^{c\boldsymbol{\tau}^{(t)}}),\quad \Delta=\{\delta_{jl}\},$$
$$\{y_r\} = \{c\tau_{jl}^{(m)}\},\quad \text{and}\quad z_r = e^{y_r},\ r=1,\dots,s,$$

$s=(t+1)n$, assuming that condition (b) is satisfied. Condition (a) of Proposition 1 holds automatically since $\delta z_r/z_r - \delta y_r = 0$ for all derivations $\delta\in\Delta$ and all $r = 1,\dots,s$; the inclusion $\delta E \subseteq E$ follows from the definition of M. Then, by Ax's theorem,

$$\operatorname{tr}\deg_M M(y_1,\dots,y_s,z_1,\dots,z_s) \geqslant s,\qquad \text{where}\quad s=(t+1)n.$$

We now recall that $M(y_1,\dots,y_s) = M(c\boldsymbol{\tau}) = M(\boldsymbol{\tau})$ and $\operatorname{tr}\deg_M M(\boldsymbol{\tau})$ is finite (more precisely, equal to n). Hence we obtain the lower estimate

$$\begin{aligned}\operatorname{tr}\deg_{M(\boldsymbol{\tau})} M(\boldsymbol{\tau}, e^{c\boldsymbol{\tau}^{(0)}}, e^{c\boldsymbol{\tau}^{(1)}},\dots,e^{c\boldsymbol{\tau}^{(t)}}) &= \operatorname{tr}\deg_{M(\boldsymbol{\tau})} M(y_1,\dots,y_s,z_1,\dots,z_s)\\ &\geqslant s - \operatorname{tr}\deg_M M(\boldsymbol{\tau}) = (t+1)n-n,\end{aligned}$$

which contradicts the previous upper estimate for sufficiently large t (that is, for $t \geqslant n$).

Thus, it is enough to produce a set $\gamma_0, \gamma_1, \dots, \gamma_t \in \Gamma = Sp_{2g}(\mathbb{Z})$ such that condition (b) of Ax's theorem is satisfied. As in the proof of [2], we shall use

$$\gamma_m = \begin{pmatrix} \mathbf{0}_g & -\mathbf{1}_g \\ \mathbf{1}_g & -m\mathbf{1}_g \end{pmatrix} \in Sp_{2g}(\mathbb{Z}), \qquad m = 0, 1, \dots, t.$$

As above, we set

$$\boldsymbol{\tau}^{(m)} = (\tau_{jl}^{(m)}) = \gamma_m(\boldsymbol{\tau}) = -(\boldsymbol{\tau} - m\mathbf{1}_g)^{-1}$$

(that is, $\tau_{jl}^{(m)} = \gamma_m \cdot \tau_{jl}$ in the notation of § 2) and suppose that condition (b) of Proposition 1 does not hold for the corresponding choice of $\{y_r\}$ and $\{z_r = e^{y_r}\}$. Then there is a non-trivial family $\{C_{jl}^{(m)}\} \in \mathbb{Z}^{nt}$ such that the rational function

$$R(\boldsymbol{\tau}) = \sum_{\substack{1 \leqslant j \leqslant l \leqslant g \\ 1 \leqslant m \leqslant t}} C_{jl}^{(m)} c\tau_{jl}^{(m)}$$

satisfies

$$e^{R(\boldsymbol{\tau})} \in M^{\mathrm{alg}},$$

where M^{alg} denotes the algebraic closure of M. Taking logarithmic derivatives, we deduce from the differential stability of M (and hence of M^{alg}) that

$$\frac{\partial R}{\partial \tau_{jl}}(\boldsymbol{\tau}) \in M^{\mathrm{alg}} \qquad \text{for all} \quad 1 \leqslant j \leqslant l \leqslant g.$$

But M^{alg} and $\mathbb{C}(\boldsymbol{\tau})$ are linearly disjoint over $\mathbb{C}$ since M and $\mathbb{C}(\boldsymbol{\tau})$ are linearly disjoint over $\mathbb{C}$ (see § 1, formula (2)) and since $\mathbb{C}(\boldsymbol{\tau})$ is a purely transcendental extension of $\mathbb{C}$. The relations obtained imply that

$$\frac{\partial R}{\partial \tau_{jl}}(\boldsymbol{\tau}) \in \mathbb{C} \qquad \text{for all} \quad 1 \leqslant j \leqslant l \leqslant g.$$

Consequently, there is a *polynomial* Q (of degree at most 1) such that

$$R = Q \in \mathbb{C}[\boldsymbol{\tau}].$$

We now consider t polynomials

$$P_m(\boldsymbol{\tau}) = \sum_{1 \leqslant j \leqslant l \leqslant g} C_{jl}^{(m)} c\tau_{jl}, \qquad m = 0, 1, \dots, t.$$

For each m, the image $R_m = \gamma_m \cdot P_m$ of P_m under the action of $\gamma_m \in \Gamma$ is the rational function

$$R_m(\boldsymbol{\tau}) = P_m\bigl(-(\boldsymbol{\tau} - m\mathbf{1}_g)^{-1}\bigr) = \sum_{1 \leqslant j \leqslant l \leqslant g} C_{jl}^{(m)} c\tau_{jl}^{(m)}, \qquad m = 0, 1, \dots, t,$$

since the action of Γ on $\mathbb{C}(\boldsymbol{\tau})$ is $\mathbb{C}$-linear. Therefore, $R = \sum_{0\leqslant m\leqslant t} R_m$, and we finally get the relation

$$\sum_{0\leqslant m\leqslant t} R_m = Q.$$

The polynomial Q may vanish identically. But since the family $\{C_{jl}^{(m)}c\} \in \mathbb{C}^{nt}$ is non-trivial, some of the polynomials P_m $(m = 0,\dots,t)$ are non-zero. These polynomials are linear forms and, in particular, have no constant terms. Therefore, the last relation contradicts the corollary of Lemma 3. This completes the proof of Theorem 1.

§ 4. Analytic proof of Theorem 2

4.1. A new proof of Theorem 1 in the case when $g = 1$. In this case, we can dispense with the differential algebra. We recall that the field M has finite transcendence degree over $\mathbb{C}$ and assume that the function $e^{c\tau}$ is algebraic over $M(\tau)$, contrary to Theorem 1. According to Lemma 1, $M(\tau)$ is stable under the action of $\Gamma = Sp_2(\mathbb{Z}) = SL_2(\mathbb{Z})$. Therefore, $e^{c\gamma(\tau)}$ is also algebraic over $M(\tau)$ for any $\gamma \in \Gamma$. We claim that

the functions $e^{c\gamma(\tau)}$ with γ running through Γ generate (over $\mathbb{C}$) a field of infinite transcendence degree over $\mathbb{C}$.

Hence, they cannot be all algebraic over the field $M(\tau)$, whose transcendence degree over $\mathbb{C}$ is finite.

Proof of the claim $(g = 1)$. The crux here is that $e^{c\tau}$ is holomorphic on the whole τ-plane, not just on the upper-half plane $\mathfrak{H}_1$. Thus, taking γ of the form $\gamma_m = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -m \end{smallmatrix}\right)$, where $m = 0,1,2,\dots$, we obtain (for each m) a function $e^{c\gamma_m(\tau)} = e^{-c/(\tau-m)}$ which is holomorphic on $\mathbb{C}$ except at the point $\tau = m$, where it has an essential singularity. But any such function f is transcendental over the field of functions which are meromorphic in a neighbourhood of m (apply the Weierstrass–Sokhotsky theorem on the density of the image of any punctured neighbourhood of m under f). Hence, for each positive integer t, the function $e^{c\gamma_t(\tau)}$ is transcendental over the field generated over $\mathbb{C}$ by the previous functions $e^{c\gamma_m(\tau)}$, $m = 0,1,\dots,t-1$, and the claim follows by induction on t.

4.2. Proof of Theorem 2. Instead of Ax's theorem of § 3, we shall use a simpler differential-algebraic result of Kolchin (see [7], Ch. VI, § 5, Exercise 4(b)). This multiplicative analogue of Ostrowski's theorem follows from the fact that any proper algebraic subgroup of a group of multiplicative type is contained in the kernel of a non-trivial character.

Proposition 2 (Kolchin's theorem). *Let $F \supseteq E \supseteq k \supseteq \mathbb{Q}$ be a tower of fields, and let Δ be a set of derivations of F such that $\delta E \subseteq E$ for all $\delta \in \Delta$ and $\bigcap_{\delta\in\Delta} \ker\delta = k$. Let $z_1,\dots,z_s$ be elements of F^* which are algebraically dependent over E. Suppose that $\delta z_r/z_r \in E$ for all $\delta \in \Delta$ and $r = 1,\dots,s$. Then there are numbers $n_1,\dots,n_s \in \mathbb{Z}$, not all equal to 0, such that $z_1^{n_1}\cdots z_s^{n_s} \in E^*$.*

We now assume that all hypotheses of Theorem 2 hold, but the functions $e^{Q_1(\boldsymbol{\tau})},\dots,e^{Q_N(\boldsymbol{\tau})}$ are algebraically dependent over $M(\boldsymbol{\tau})$ contrary to the conclusion. Applying Proposition 2 to the $\boldsymbol{\delta}$-differential field $E = M(\boldsymbol{\tau})$ and the functions

$z_r = e^{Q_r(\boldsymbol{\tau})}$ ($r = 1, \dots, s$ with $s := N$), we deduce that there is a non-trivial family $\{c_1, \dots, c_N\} \in \mathbb{Z}^N$ such that the polynomial

$$P(\boldsymbol{\tau}) = \sum_{1 \leqslant r \leqslant N} c_r Q_r(\boldsymbol{\tau})$$

satisfies

$$e^{P(\boldsymbol{\tau})} \in M(\boldsymbol{\tau}).$$

Since the polynomials Q_r are linearly independent over $\mathbb{Q}$ and have no constant terms, P is a non-zero polynomial with no constant term. By Lemma 1, it follows that the function $e^{(\gamma \cdot P)(\boldsymbol{\tau})} = e^{P(\gamma(\boldsymbol{\tau}))}$ lies in $M(\boldsymbol{\tau})$ for any $\gamma \in \Gamma$.

We now consider the elements of Γ of the form

$$\gamma_m = \begin{pmatrix} \mathbf{0}_g & -\mathbf{1}_g \\ \mathbf{1}_g & -m\mathbf{1}_g \end{pmatrix} \in Sp_{2g}(\mathbb{Z}), \qquad m = 0, 1, \dots .$$

We claim that

the functions $e^{\gamma_m \cdot P}$, $m = 0, \dots, t$, are algebraically independent over $\mathbb{C}$ for any positive integer t.

Consequently, if t is sufficiently large (more precisely, if $t > \frac{1}{2}g(5g+3)$ according to § 1, formula (2)), then not all these functions can lie in the field $M(\boldsymbol{\tau})$, which has finite transcendence degree over $\mathbb{C}$. This contradiction will complete the proof of Theorem 2.

Proof of the claim ($g \geqslant 1$). Writing the polynomial P as the sum of its non-zero homogeneous parts of degree $\geqslant 1$, we deduce from Lemma 3 and the factoriality of $\mathbb{C}[\boldsymbol{\tau}]$ that the polar divisor in Z_g of the rational function

$$R_m(\boldsymbol{\tau}) = \gamma_m \cdot P(\tau) = P\bigl(-(\boldsymbol{\tau} - m\mathbf{1}_g)^{-1}\bigr), \qquad m = 0, 1, \dots, t,$$

is set-theoretically equal to the divisor $\mathcal{D}_m$. We now fix a point $\tau' \in \mathcal{D}_t$ that does not belong to any of the polar divisors of the functions $R_m = \gamma_m \cdot P$, $m = 0, 1, \dots, t-1$, (this is possible by the corollary of Lemma 2), nor to the zero divisor of R_t (that is, to the set of points of indeterminacy of R_t), nor to the singular locus of $\mathcal{D}_t$. Then the functions $e^{R_m(\boldsymbol{\tau})}$, $m = 0, 1, \dots, t-1$, lie in the field $\mathfrak{M}_{\boldsymbol{\tau}'}$ of germs of meromorphic functions at $\boldsymbol{\tau}'$, and it remains to prove that $f = e^{R_t}$ cannot satisfy a non-trivial algebraic equation $S(f) = 0$ over $\mathfrak{M}_{\boldsymbol{\tau}'}$.

Consider an analytic curve $C \subset Z_g(\mathbb{C})$ such that C intersects $\mathcal{D}_t$ transversally at $\boldsymbol{\tau}'$ and the germ of C at $\boldsymbol{\tau}'$ is not contained in any of the zero or polar divisors of the coefficients of S. Let

$$\phi\colon \{z \in \mathbb{C} : |z| < 1\} \to C, \qquad \phi(0) = \boldsymbol{\tau}',$$

be a parametrization of C. By the transversality and the non-indeterminacy of R_t at $\boldsymbol{\tau}'$, the one-variable function $f \circ \phi(z)$ has an essential singularity at 0 while the pullbacks of the coefficients of S under ϕ are the germs at 0 of well-defined non-zero meromorphic functions of z. As mentioned in § 4.1, the resulting algebraic dependence relation for $f \circ \phi$ is absurd, and our claim follows.

Remark 2. Applying Proposition 2 twice, we can easily transform the arguments of this section into a purely algebraic proof of Theorem 2. Indeed, analysis occurred only in the proof of the claim. But if the claim does not hold, we can apply Kolchin's theorem again (this time with $E = \mathbb{C}(\boldsymbol{\tau})$) and use the non-exactness of non-zero logarithmic differentials on the affine space Z_g to conclude that the rational functions $R_m(\boldsymbol{\tau})$, $m = 0, 1, \dots, t$, are linearly dependent over $\mathbb{Z}$ modulo $\mathbb{C}$. This contradicts the corollary of Lemma 3 applied to $P_0 = \dots = P_t := P$ and $Q = 1$.

Remark 3. In a similar vein, the deduction of Theorem 1 from Ax's theorem in § 3 used the corollary of Lemma 3 only when $P_0, \dots, P_t$ are linear forms. (This case admits a much simpler proof, which requires Lemma 2 only.) We leave it to the reader to verify that the arguments of § 3 can be transformed into a proof of Theorem 2 if we apply this corollary in its full generality. (Throughout § 3, replace the n linear forms τ_{jl} by the N polynomials Q_r of Theorem 2 and replace $\tau_{jl}^{(m)} = \gamma_m \cdot \tau_{jl}$ by $\gamma_m \cdot Q_r$.)

Remark 4. The most significant difference between the proofs in §§ 3, 4 is that they use different arguments from [3]: in § 3 we merely need to know that $M(\boldsymbol{\tau})$ is a purely transcendental extension of M, while the arguments of § 4 rely solely on the finiteness of the transcendence degree of M over $\mathbb{C}$. (Of course, the stability of $M(\boldsymbol{\tau})$ under Γ is the main argument in both cases.)

§ 5. Thetanulls and their logarithmic derivatives

Setting

$$q_{jl} = e^{2\pi i \tau_{jl}}, \quad 1 \leqslant j < l \leqslant g, \qquad \text{and} \qquad q_{jj} = e^{\pi i \tau_{jj}}, \quad 1 \leqslant j \leqslant g,$$

we now turn to modular forms and consider the *thetanulls*,

$$\vartheta_{\boldsymbol{a}} = \vartheta_{(\boldsymbol{a}', \boldsymbol{a}'')}(\boldsymbol{q}) = \sum_{\boldsymbol{n} \in \mathbb{Z}^g} e^{\pi i^t(\boldsymbol{n} + \boldsymbol{a}'/2)\boldsymbol{a}''} \prod_{1 \leqslant j \leqslant l \leqslant g} q_{jl}^{(n_j + a_j'/2)(n_l + a_l'/2)},$$

attached to even 2-characteristics $\boldsymbol{a} = (\boldsymbol{a}', \boldsymbol{a}'') \in \mathfrak{K}_+ \subset (\mathbb{Z}/2\mathbb{Z})^{2g}$, that is, characteristics with ${}^t\boldsymbol{a}' \cdot \boldsymbol{a}'' \equiv 0 \pmod 2$. These series converge in a non-trivial domain of $\boldsymbol{q}$-space, where $\boldsymbol{q} = \{q_{jl},\ 1 \leqslant j \leqslant l \leqslant g\} \in \mathbb{C}^n$, but we shall regard them as formal elements in the ring

$$\mathfrak{F}_{\boldsymbol{q}} = \mathbb{C}[[\boldsymbol{q}^{\boldsymbol{\nu}},\ 4\boldsymbol{\nu} \in Z_g(\mathbb{Z}),\ \boldsymbol{\nu} \geqslant 0]].$$

(To recover the standard Fourier expansions, put $\boldsymbol{q}^{\boldsymbol{\nu}} = e^{i\pi \operatorname{Tr}(\boldsymbol{\nu}\boldsymbol{\tau})}$.)

The set of partial derivations $\boldsymbol{\delta}$ is now transformed into

$$\delta_{jl} = q_{jl} \frac{\partial}{\partial q_{jl}}, \qquad 1 \leqslant j \leqslant l \leqslant g,$$

which makes $\mathfrak{F}_{\boldsymbol{q}}$ into a differential ring. We define the *logarithmic derivatives of thetanulls* by

$$\psi_{\boldsymbol{a},jl} = \frac{\delta_{jl}\vartheta_{\boldsymbol{a}}}{\vartheta_{\boldsymbol{a}}}, \qquad \boldsymbol{a} \in \mathfrak{K}_+, \quad 1 \leqslant j \leqslant l \leqslant g.$$

Since

$$\vartheta_{\boldsymbol{a}} = 1 + \sum_{\boldsymbol{0}\neq\boldsymbol{n}\in\mathbb{Z}^g} (-1)^{{}^t\boldsymbol{n}\boldsymbol{a}''} \prod_{1\leqslant j\leqslant l\leqslant g} q_{jl}^{n_j n_l} \qquad \text{if} \quad \boldsymbol{a}' = 0, \tag{3}$$

the n logarithmic derivatives of the corresponding 2^g functions

$$\psi_{\boldsymbol{a},pq} = \sum_{\boldsymbol{0}\neq\boldsymbol{n}\in\mathbb{Z}^g} (-1)^{{}^t\boldsymbol{n}\boldsymbol{a}''} n_p n_q \prod_{1\leqslant j\leqslant l\leqslant g} q_{jl}^{n_j n_l} \cdot \sum_{m=0}^{\infty} \Biggl(- \sum_{\boldsymbol{0}\neq\boldsymbol{n}\in\mathbb{Z}^g} (-1)^{{}^t\boldsymbol{n}\boldsymbol{a}''} \prod_{1\leqslant j\leqslant l\leqslant g} q_{jl}^{n_j n_l} \Biggr)^m$$

$(1 \leqslant p \leqslant q \leqslant g)$ lie in $\mathfrak{F}_{\boldsymbol{q}}$, while the logarithmic derivatives of the other thetanulls $(\boldsymbol{a}' \neq 0)$ belong to the field of fractions of $\mathfrak{F}_{\boldsymbol{q}}$.

According to [3], § 5, the field of fractions of the ring

$$Q_g = \mathbb{Q}[\vartheta_{\boldsymbol{a}}, \psi_{\boldsymbol{a},jl}]_{\boldsymbol{a}\in\mathfrak{K}_+;1\leqslant j\leqslant l\leqslant g}$$

is $\boldsymbol{\delta}$-stable, and its algebraic closure coincides with that of M. Therefore Theorem 1 yields the following theorem.

Theorem 3. *The field of fractions of the ring*

$$\Pi_g = \mathbb{Q}[q_{jl}, \vartheta_{\boldsymbol{a}}, \psi_{\boldsymbol{a},jl}]_{\boldsymbol{a}\in\mathfrak{K}_+;1\leqslant j\leqslant l\leqslant g}$$

is stable under the derivations $\boldsymbol{\delta}$, *and has transcendence degree* $\frac{1}{2}g(5g+3)$ *over* $\mathbb{Q}$.

In the classical case $g = 1$, as well as in the case $g = 2$, the *ring* Q_g is stable under the derivations $\boldsymbol{\delta}$ (see [3], § 6). Hence the rings Π_1 and Π_2 have the same property. Furthermore, if $g = 1$, then the product formulae for the thetanulls ϑ_a (see, for example, [8], § 21.42) provide explicit expressions for the q-expansions of their logarithmic derivatives ψ_a. Recent results of Borcherds show that there is some analogy between the classical case and the case $g = 2$ in this respect too. We conclude this paper by explaining how Borcherds' formulae provide the explicit $\boldsymbol{q}$-expansions of the 10 elements which form (according to [3], Theorem 3 (iv)) a transcendence basis of Q_2 over $\mathbb{Q}$ and hence (by Theorem 3) also of Π_2 over $\mathbb{Q}[\boldsymbol{q}]$.

We henceforth assume that $g = 2$. For simplicity, we use (as in [3], § 6) the map $(\mathbb{Z}/2\mathbb{Z})^2 \to \{0,1,2,3\}$ with

$$(0,0) \mapsto 0, \quad (0,1) \mapsto 1, \quad (1,0) \mapsto 2, \quad (1,1) \mapsto 3$$

to represent a characteristic $\boldsymbol{a} = (\boldsymbol{a}', \boldsymbol{a}'') \in (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2$ by only two digits. Then $\mathfrak{K}_+ = \{00, 01, 02, 03, 10, 12, 20, 21, 30, 33\}$. We relabel the entries of $\boldsymbol{q}$ by setting $q_1 = q_{11}$, $q_2 = q_{22}$, $q_3 = q_{12}$ and do the same for the derivations

$$\delta_j = q_j \frac{\partial}{\partial q_j}, \qquad j = 1,2,3,$$

and the logarithmic derivatives $\psi_{\boldsymbol{a},j}$. In this notation, the transcendence basis for Q_2 mentioned above is given by the list

$$\vartheta_{00}, \vartheta_{01}, \vartheta_{02}, \ \psi_{00,1}, \psi_{01,1}, \psi_{02,1}, \ \psi_{00,2}, \psi_{01,2}, \psi_{02,2}, \ \psi_{00,3}. \tag{4}$$

Finally, we denote by Z_2^+ the space of positive semi-definite symmetric matrices $\boldsymbol{\nu} = \left(\begin{smallmatrix} \nu_1 & \nu_3 \\ \nu_3 & \nu_2 \end{smallmatrix}\right)$ with entries $\nu_1, \nu_2, 2\nu_3 \in \mathbb{Z}$, for which we again set $\boldsymbol{q}^{\boldsymbol{\nu}} = q_1^{\nu_1} q_2^{\nu_2} q_3^{\nu_3}$. Then we have the following proposition (see [9], § 4).

Proposition 3 (Borcherds' product formula). *The following identity holds*:

$$\vartheta_{03}(\boldsymbol{q}) = \sum_{n_1,n_2\in\mathbb{Z}} (-1)^{n_1+n_2} q_1^{n_1^2} q_2^{n_2^2} q_3^{n_1 n_2} = \prod_{\mathbf{0}_2\neq\boldsymbol{\nu}\in Z_2^+} \left(\frac{1-\boldsymbol{q}^{\boldsymbol{\nu}}}{1+\boldsymbol{q}^{\boldsymbol{\nu}}}\right)^{f(\det\boldsymbol{\nu})}, \tag{5}$$

where

$$\sum_{m=0}^{\infty} f(m) q^m = \left(\sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}\right)^{-1} = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + 24q^5 + \cdots$$

is the reciprocal of the one-variable thetanull $\vartheta_{01}(q)$.

Substituting $q_1 \mapsto -q_1$ and $q_2 \mapsto -q_2$ in (5) and (3), we derive the product formulae for the other three thetanulls with $\boldsymbol{a}' = 0$:

$$\vartheta_{01} = \prod_{\mathbf{0}_2\neq\boldsymbol{\nu}\in Z_2^+} \left(\frac{1-(-1)^{\nu_1}\boldsymbol{q}^{\boldsymbol{\nu}}}{1+(-1)^{\nu_1}\boldsymbol{q}^{\boldsymbol{\nu}}}\right)^{f(\det\boldsymbol{\nu})}, \quad \vartheta_{02} = \prod_{\mathbf{0}_2\neq\boldsymbol{\nu}\in Z_2^+} \left(\frac{1-(-1)^{\nu_2}\boldsymbol{q}^{\boldsymbol{\nu}}}{1+(-1)^{\nu_2}\boldsymbol{q}^{\boldsymbol{\nu}}}\right)^{f(\det\boldsymbol{\nu})},$$
$$\vartheta_{00} = \prod_{\mathbf{0}_2\neq\boldsymbol{\nu}\in Z_2^+} \left(\frac{1-(-1)^{\nu_1+\nu_2}\boldsymbol{q}^{\boldsymbol{\nu}}}{1+(-1)^{\nu_1+\nu_2}\boldsymbol{q}^{\boldsymbol{\nu}}}\right)^{f(\det\boldsymbol{\nu})}. \tag{6}$$

Explicit formulae for the $\boldsymbol{q}$-expansions of the logarithmic derivatives of these thetanulls follow from (5), (6) in the obvious way. For instance, if $\boldsymbol{a}'' = 3$ and $j = 1, 2, 3$, then we get

$$\psi_{03,j} = \frac{\delta_j \vartheta_{03}}{\vartheta_{03}} = -2 \sum_{\mathbf{0}_2\neq\boldsymbol{\nu}\in Z_2^+} \frac{\nu_j f(\det\boldsymbol{\nu})\boldsymbol{q}^{\boldsymbol{\nu}}}{1-\boldsymbol{q}^{2\boldsymbol{\nu}}}$$
$$= -2 \sum_{\mathbf{0}_2\neq\boldsymbol{\mu}\in Z_2^+} \boldsymbol{q}^{\boldsymbol{\mu}} \left(\sum_{\substack{\boldsymbol{\nu}\in Z_2^+:(2m+1)\boldsymbol{\nu}=\boldsymbol{\mu} \\ m\in\mathbb{Z},m\geqslant 0}} \nu_j f(\det\boldsymbol{\nu}) \right).$$

Combining (3) and (6), we similarly get the promised $\boldsymbol{q}$-expansions of all the elements listed in (4).

Remark 5. In genus 2, product expansions also exist for thetanulls other than the four studied above. We mention for instance ϑ_{33} (see [10], Example 2.3, where this thetanull is dubbed "most odd"). See also Example 2.4 of [10] for the product expansion of the modular form of weight 5 given by $\prod_{\boldsymbol{a}\in\mathfrak{K}_+} \vartheta_{\boldsymbol{a}}$.

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