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Cancellation of factorials

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Abstract. An arithmetical property allowing an improvement of some number-theoretic estimates is studied. Previous results were mostly qualitative. Application of quantitative results of the paper to the class of generalized hypergeometric G-functions extends the set of irrational numbers representable as values of these functions.

Bibliography: 20 titles.

1. Introduction. The term used in the title is of recent origin, although some effects of the phenomenon of 'cancellation of factorials' are already classical and included in lists of problems for students.

Example 1. The operator $D = d/dz$ of differentiation with respect to the variable z takes the ring $\mathbb{Z}[z]$ of polynomials with integer coefficients into itself; hence the operators D^n, $n = 0, 1, 2, \dots$, also have this property. It is easy to show (see, for instance, [1], Chapter 4, Lemma 7) that the operators $D^n/n!$ also take $\mathbb{Z}[z]$ into itself:

$$\frac{1}{n!}D^n = \frac{1}{n!}\frac{d^n}{dz^n} : \mathbb{Z}[z] \to \mathbb{Z}[z], \qquad n = 0, 1, 2, \dots .$$

Example 2. The sequence of polynomials

$$\langle\lambda\rangle_0 = 1, \qquad \langle\lambda\rangle_n = \lambda(\lambda-1)\cdots(\lambda-n+1), \quad n = 1, 2, \dots, \tag{1}$$

lies in the ring $\mathbb{Z}[\lambda]$; hence the polynomials (1) take integer values for integer λ. In fact, this property is exhibited already by the polynomials

$$\Delta_n(\lambda) = \frac{\langle\lambda\rangle_n}{n!}, \qquad n = 0, 1, 2, \dots \tag{2}$$

(see, for instance, [2], Part 8, Chapter 2, Problem 84), which for $n \geqslant 2$ do not belong to $\mathbb{Z}[\lambda]$. The polynomials (2) are said to be *integral-valued.*

Let $\mathbb{K}$ be a field that is an algebraic extension of the field $\mathbb{Q}$ of rationals and $\mathbb{Z}_{\mathbb{K}}$ its ring of integers. In place of $\mathbb{Z}[z]$ in Example 1 we can consider the ring $\mathbb{Z}_{\mathbb{K}}[z]$:

$$\frac{1}{n!}\frac{d^n}{dz^n} : \mathbb{Z}_{\mathbb{K}}[z] \to \mathbb{Z}_{\mathbb{K}}[z], \qquad n = 0, 1, 2, \dots .$$

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Definition 1. An *operator* $D\colon \mathbb{K}[z] \to \mathbb{K}[z]$ is said to have the property of the *cancellation of factorials with constant* $\Psi \geqslant 1$ if there exists a sequence of positive integers $\{\psi_k\}_{k\in\mathbb{N}}$ such that

$$\psi_k \frac{D^n}{n!}\colon \mathbb{Z}_{\mathbb{K}}[z] \to \mathbb{Z}_{\mathbb{K}}[z], \qquad n = 0, 1, \dots, k, \quad k \in \mathbb{N}, \qquad \text{and} \qquad \varlimsup_{k\to\infty} \psi_k^{1/k} \leqslant \Psi.$$

Thus, we have the following result.

Lemma 1. *For each algebraic extension* $\mathbb{K}$ *of* $\mathbb{Q}$ *the operator* $d/dz\colon \mathbb{K}[z] \to \mathbb{K}[z]$ *has the property of the cancellation of factorials with constant* 1.

Taking in Example 2 $\lambda = a/b \in \mathbb{Q}$ with coprime $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ we see that the common denominator of the quantities

$$b^n \frac{\langle\lambda\rangle_n}{n!} = \frac{a(a-b)(a-2b)\cdots(a-(n-1)b)}{n!}, \qquad n = 1, 2, \dots, k, \tag{3}$$

is $\prod_{p|b} p^{\tau_p(k)}$ (see [1], Chapter 1, Lemma 8, or [3], Chapter I, Appendix), where

$$\tau_p(k) = \left\lfloor \frac{k}{p} \right\rfloor + \left\lfloor \frac{k}{p^2} \right\rfloor + \left\lfloor \frac{k}{p^3} \right\rfloor + \dots \leqslant \frac{k}{p-1} \tag{4}$$

is the power of the prime p in the factorization of $k!$; here and throughout, $\lfloor\,\cdot\,\rfloor$ denotes the integer part of a number. Thus, the common denominator of the quantities in (3) has the estimate $e^{k\chi(b)}$, where

$$\chi(b) = \sum_{p|b} \frac{\log p}{p-1}, \qquad b \in \mathbb{N}, \tag{5}$$

that is, has a geometric order of growth as $k \to \infty$.

Of course, for the domain of the polynomials (1) we can take an algebraic extension $\mathbb{K}$ of the field $\mathbb{Q}$ (or even the ring of square matrices with entries in $\mathbb{K}$, see § 4 below). From the outset we define the *denominator* $\operatorname{den}\lambda$ of $\lambda \in \mathbb{K}$ as the least positive integer b such that $b\lambda \in \mathbb{Z}_{\mathbb{K}}$.

Definition 2. We say that an *element* λ *of the field* $\mathbb{K}$ *has the property of the cancellation of factorials with constant* $\Psi \geqslant 1$ if there exists a sequence of positive integers $\{\psi_k\}_{k\in\mathbb{N}}$ such that

$$\psi_k \frac{\langle\lambda\rangle_n}{n!} \in \mathbb{Z}_{\mathbb{K}}, \qquad n = 0, 1, \dots, k, \quad k \in \mathbb{N}, \qquad \text{and} \qquad \varlimsup_{k\to\infty} \psi_k^{1/k} \leqslant \Psi,$$

where the symbol $\langle\,\cdot\,\rangle_n$ is defined in (1).

We summarize our comments relating to Example 2 in the following statement.

Lemma 2. *A rational number λ has the property of the cancellation of factorials with constant $be^{\chi(b)}$, where $b = \operatorname{den}\lambda$ and the function $\chi(\,\cdot\,)$ is defined in* (5).

Remark. It is essential in Lemma 2 that λ be rational. By considering a field $\mathbb{K} = \mathbb{Q}(\lambda)$ of finite degree $\varkappa = [\mathbb{K} : \mathbb{Q}] \geqslant 2$ and setting in the main lemma of [4] $a_1 = -\lambda - 1$, $b_1 = 0$, $m = m_1 = 1$, $m_2 = 0$, $\tau = 1 - 1/\varkappa$, and $\varepsilon = 1/6$ we obtain the following result: *for each sequence of positive integers $\{\psi_k\}_{k\in\mathbb{N}}$ such that*

$$\psi_k \frac{\langle\lambda\rangle_n}{n!} \in \mathbb{Z}_{\mathbb{K}}, \qquad n = 0, 1, \dots, k, \quad k \in \mathbb{N},$$

the inequality

$$\psi_k \geqslant Ck^{(\tau-\varepsilon)\varkappa k} \geqslant Ck^{2k/3}, \qquad k \in \mathbb{N}, \tag{6}$$

holds with positive constant C depending only on λ. The estimate (6) means that for irrational λ the growth of common denominators of the sequence (2) as $k \to \infty$ is at least factorial and a discussion of the cancellation of factorials would be pointless.

2. History of the problem. The examples of § 1 can be considered classical. In fact, the concept of *cancellation of factorials* appeared for the first time in a paper by Galochkin [5], in connection with the cancellation of coefficients of linear approximating forms (Padé approximants) for so-called G-functions in the Siegel–Shidlovskiĭ method. Allusions to this phenomenon can be found in Siegel's paper [6], which is in effect the origin of this method. All authors who applied the Siegel–Shidlovskiĭ method to G-functions before [5] obtained estimates of linear forms and of polynomials in the values of these functions at rational points whose absolute values depended on the height of the forms under consideration (see, for instance, [7]). And it was the use of the cancellation of factorials that enabled one to obtain, for a subclass of G-functions, estimates of the moduli of polynomials in the values of these functions at points of absolute value independent of the height of the polynomial. A statement of a theorem on an effective estimate of a linear form of G-functions from some subclass appeared in [5]; its proof was published in [8]. In [9] the authors calculate a constant of the cancellation of factorials for the Gauss hypergeometric function, which made it possible to obtain several results on the irrationality and the linear independence of the values of this function and its derivative. Finally, in 1985 D. and G. Chudnovsky proved [10] that the condition for the cancellation of factorials formulated in [5] is fulfilled by the homogeneous systems of linear differential equations which hold for the G-functions. In [3], Chapter VI, § 4, an easy generalization of the Chudnovskys' construction allowed the author to extend this property to inhomogeneous systems of linear differential equations.

Despite the affirmative solution of the problem of the cancellation of factorials for G-functions, the known values of the corresponding constant are rather crude. This is because the Chudnovskys' approach uses implicit constructions based on the Dirichlet principle. In the present paper we give a new solution of the same problem for the generalized hypergeometric G-functions, the possibility of which was conjectured in [11], § 12. Our proof is based on explicit constructions and therefore produces a better estimate of the constant of the cancellation of factorials. Number theoretic applications of this result extend the set of irrational numbers representable as values of hypergeometric G-functions.

We now define a class of G-functions having a Taylor expansion with rational coefficients in a neighbourhood of the origin. We say that a *family of functions*

$$f_j(z)=\sum_{n=0}^{\infty} f_{jn}z^n,\quad j=1,\dots,m,\qquad f_{jn}\in\mathbb{Q},\quad j=1,\dots,m,\quad n=0,1,\dots,\tag{7}$$

belongs to the class $\mathbf{G}(C,\Phi)$ if the functions (7) are analytic in the disc $|z|<C$ and there exists a sequence of positive integers $\{\varphi_k\}_{k\in\mathbb{N}}$ such that

$$\varphi_k f_{jn}\in\mathbb{Z},\qquad j=1,\dots,m,\quad n=0,1,\dots,k,\quad k\in\mathbb{N},\quad\text{and}\quad \varlimsup_{k\to\infty}\varphi_k^{1/k}\leqslant\Phi.$$

We now formulate the property of the cancellation of factorials for systems of linear differential equations

$$\begin{gathered}\frac{d}{dz}y_l=Q_{l0}+\sum_{j=1}^{m}Q_{lj}y_j,\qquad l=1,\dots,m,\\ Q_{lj}=Q_{lj}(z)\in\mathbb{Q}(z),\qquad l=1,\dots,m,\quad j=0,\dots,m,\end{gathered}\tag{8}$$

which hold for the G-functions (7).

Let $T(z)\in\mathbb{Q}[z]$ be the denominator of the rational functions Q_{lj} with leading coefficient 1:

$$T(z)Q_{lj}(z)\in\mathbb{Q}[z],\qquad l=1,\dots,m,\quad j=0,\dots,m.\tag{9}$$

It follows from (8) that for the derivatives of order n, $n=1,2,\dots$, we have

$$\begin{gathered}\frac{d^n}{dz^n}y_l=Q_{l0}^{[n]}+\sum_{j=1}^{m}Q_{lj}^{[n]}y_j,\qquad l=1,\dots,m,\\ Q_{lj}^{[n]}=Q_{lj}^{[n]}(z)\in\mathbb{Q}(z),\qquad l=1,\dots,m,\quad j=0,\dots,m.\end{gathered}\tag{10}$$

Easy calculations demonstrate the following recurrence relations:

$$\begin{gathered}Q_{lj}^{[n]}(z)=\frac{d}{dz}Q_{lj}^{[n-1]}(z)+\sum_{r=1}^{m}Q_{lr}^{[n-1]}(z)Q_{rj}(z),\\ l=1,\dots,m,\quad j=0,1,\dots,m,\quad n=1,2,\dots;\end{gathered}\tag{11}$$

therefore

$$\begin{gathered}T^n(z)Q_{lj}^{[n]}(z)=T(z)\frac{d}{dz}\bigl(T^{n-1}(z)Q_{lj}^{[n-1]}(z)\bigr)-(n-1)T'(z)\cdot T^{n-1}(z)Q_{lj}^{[n-1]}(z)\\ +\sum_{r=1}^{m}T^{n-1}(z)Q_{lr}^{[n-1]}(z)\cdot T(z)Q_{rj}(z),\\ l=1,\dots,m,\quad j=0,1,\dots,m,\quad n=1,2,\dots.\end{gathered}$$

Hence, in particular,

$$T^n(z)Q_{lj}^{[n]}(z)\in\mathbb{Q}[z],\qquad l=1,\dots,m,\quad j=0,\dots,m,\quad n=1,2,\dots.$$

Definition 3. A *system of linear differential equations* (8) is said to *have the property of the cancellation of factorials with constant* $\Psi \geqslant 1$ if there exist positive integers $\{\psi_k\}_{k\in\mathbb{N}}$ such that

$$\psi_k \frac{T^n(z)Q_{lj}^{[n]}(z)}{n!} \in \mathbb{Z}[z], \quad l=1,\dots,m, \quad j=0,\dots,m, \quad n=0,1,\dots,k, \quad k\in\mathbb{N},$$
$$\varlimsup_{k\to\infty} \psi_k^{1/k} \leqslant \Psi.$$

This property of the cancellation of factorials for a system (8) is known to hold, generally speaking, only if there exists a family of G-functions linearly independent over $\mathbb{C}(z)$ and solving this system. Such systems lie in the class of systems of differential equations of Fuchs type. Up to a meromorphic transformation of the solution space the coefficient matrix $Q(z) = \bigl(Q_{lj}(z)\bigr)_{l,j}$ of a Fuchs-type system (8) has the following form:

$$Q(z) = \frac{1}{z-\gamma_1}A_1 + \dots + \frac{1}{z-\gamma_s}A_s, \tag{12}$$

where $\gamma_1,\dots,\gamma_s$ are the regular singularities of the system (8) and $A_1,\dots,A_s$ are constant matrices (see [12], the remark to § 2.4). In the case (12) the denominator of the corresponding system (8) is $T(z) = (z-\gamma_1)\cdots(z-\gamma_s)$.

3. Cancellation of factorials for differential operators. In this section we study a generalization of the operator of differentiation d/dz that also has the property of the cancellation of factorials.

Let $\mathbb{K}$ be an algebraic extension of the field $\mathbb{Q}$ and let

$$D = \frac{d}{dz} + \frac{\lambda}{z}, \qquad \lambda\in\mathbb{Q}. \tag{13}$$

The operator $T(z)D$, where $T(z) = z$, takes the ring $\mathbb{K}[z]$ into itself, whereas D on its own takes $\mathbb{K}[z]$ into $\mathbb{K}(z)$. For this reason we extend slightly the domain of application of Definition 1.

Definition 1′. By the *denominator* of an arbitrary (not necessarily linear) differential operator $D\colon \mathbb{K}(z)\to\mathbb{K}(z)$ we shall mean the non-trivial polynomial $T(z)\in\mathbb{K}[z]$ of lowest degree with leading coefficient 1 such that $T(z)D\colon \mathbb{K}[z]\to\mathbb{K}[z]$. We say that *the operator D has the property of the cancellation of factorials with constant* $\Psi\geqslant 1$ if there exists a sequence of positive integers $\{\psi_k\}_{k\in\mathbb{N}}$ such that

$$\psi_k \frac{T^n(z)D^n}{n!}\colon \mathbb{Z}_{\mathbb{K}}[z]\to\mathbb{Z}_{\mathbb{K}}[z], \qquad n=0,1,\dots,k, \quad k\in\mathbb{N}, \quad \text{and} \quad \varlimsup_{k\to\infty}\psi_k^{1/k}\leqslant\Psi.$$

Lemma 3. *The following identities hold for the differential operator* (13):

$$D^n = \sum_{l=0}^{n} \binom{n}{l} \frac{\langle\lambda\rangle_l}{z^l} \frac{d^{n-l}}{dz^{n-l}}, \qquad n\in\mathbb{N}, \tag{14}$$

where the symbol $\langle\,\cdot\,\rangle_l$ is defined in (1).

Proof. For $n=1$ formula (14) coincides with the definition (13) of the operator D. Assuming that (14) holds for some positive integer n, we obtain

$$\begin{aligned} D^{n+1} &= \left(\frac{d}{dz}+\frac{\lambda}{z}\right)D^n = \frac{d}{dz}\sum_{l=0}^{n}\binom{n}{l}\frac{\langle\lambda\rangle_l}{z^l}\frac{d^{n-l}}{dz^{n-l}} + \frac{\lambda}{z}\sum_{l=0}^{n}\binom{n}{l}\frac{\langle\lambda\rangle_l}{z^l}\frac{d^{n-l}}{dz^{n-l}} \\ &= \sum_{l=0}^{n}\binom{n}{l}\frac{\langle\lambda\rangle_l}{z^l}\frac{d^{n-l+1}}{dz^{n-l+1}} - \sum_{l=0}^{n}\binom{n}{l}\frac{l\cdot\langle\lambda\rangle_l}{z^{l+1}}\frac{d^{n-l}}{dz^{n-l}} + \sum_{l=0}^{n}\binom{n}{l}\frac{\lambda\cdot\langle\lambda\rangle_l}{z^{l+1}}\frac{d^{n-l}}{dz^{n-l}} \\ &= \sum_{l=0}^{n}\binom{n}{l}\frac{\langle\lambda\rangle_l}{z^l}\frac{d^{n+1-l}}{dz^{n+1-l}} + \sum_{l=0}^{n}\binom{n}{l}\frac{\langle\lambda\rangle_{l+1}}{z^{l+1}}\frac{d^{n-l}}{dz^{n-l}} \\ &= \sum_{l=0}^{n}\binom{n}{l}\frac{\langle\lambda\rangle_l}{z^l}\frac{d^{n+1-l}}{dz^{n+1-l}} + \sum_{l=1}^{n+1}\binom{n}{l-1}\frac{\langle\lambda\rangle_l}{z^l}\frac{d^{n+1-l}}{dz^{n+1-l}} \\ &= \sum_{l=0}^{n+1}\binom{n+1}{l}\frac{\langle\lambda\rangle_l}{z^l}\frac{d^{n+1-l}}{dz^{n+1-l}}. \end{aligned}$$

Thus, (14) holds also for $n+1$. By the principle of mathematical induction it holds for all positive integers n. The proof is complete.

Using the definition of binomial coefficients, from Lemma 3 we deduce the identity

$$z^n\frac{D^n}{n!} = \sum_{l=0}^{n} z^{n-l}\cdot\frac{\langle\lambda\rangle_l}{l!}\cdot\frac{1}{(n-l)!}\frac{d^{n-l}}{dz^{n-l}}, \qquad n\in\mathbb{N}.$$

Applying now the results on the cancellation of factorials for the operator d/dz (Lemma 1) and the rational number λ (Lemma 2) we obtain the following statement.

Theorem 1. *The differential operator* (13) *has the property of the cancellation of factorials with constant* $be^{\chi(b)}$, *where* $b=\operatorname{den}\lambda$ *and the function* $\chi(\,\cdot\,)$ *is defined by* (5).

Lemma 3 is easily extended to the case of the differential operator

$$D=\frac{d}{dz}+\frac{\lambda_1}{z-\gamma_1}+\dots+\frac{\lambda_s}{z-\gamma_s}, \qquad \lambda_1,\dots,\lambda_s,\gamma_1,\dots,\gamma_s\in\mathbb{Q} \tag{15}$$

with denominator $T(z)=(z-\gamma_1)\cdots(z-\gamma_s)$. We indicate the corresponding identities without proofs.

Lemma 4. *The following identities hold for the differential operator* (15):

$$D^n=\sum_{\substack{n_0,n_1,\dots,n_s\geqslant0\\ n_0+n_1+\dots+n_s=n}}\frac{n!}{n_0!\,n_1!\cdots n_s!}\frac{\langle\lambda_1\rangle_{n_1}}{(z-\gamma_1)^{n_1}}\cdots\frac{\langle\lambda_s\rangle_{n_s}}{(z-\gamma_s)^{n_s}}\frac{d^{n_0}}{dz^{n_0}}, \quad n\in\mathbb{N}. \tag{16}$$

By Lemma 4,

$$\begin{aligned}\frac{T^n(z)D^n}{n!} = \sum_{\substack{n_0,n_1,\dots,n_s\geqslant0\\ n_0+n_1+\dots+n_s=n}} &(z-\gamma_1)^{n-n_1}\cdots(z-\gamma_s)^{n-n_s} \\ &\times\frac{\langle\lambda_1\rangle_{n_1}}{n_1!}\cdots\frac{\langle\lambda_s\rangle_{n_s}}{n_s!}\cdot\frac{1}{n_0!}\frac{d^{n_0}}{dz^{n_0}}, \qquad n\in\mathbb{N},\end{aligned}$$

and therefore the differential operator (15) also has the property of the cancellation of factorials.

Theorem 2. *The differential operator* (15) *has the property of the cancellation of factorials with constant* $qbe^{\chi(b)}$, *where* q *is the product of the denominators of the numbers* $\gamma_1, \dots, \gamma_s$, $b = \operatorname{den}(\lambda_1, \dots, \lambda_s)$ *is the least common denominator of* $\lambda_1, \dots, \lambda_s$, *and the function* $\chi(\,\cdot\,)$ *is defined by* (5).

4. Cancellation of factorials for square matrices. A constant square number matrix A of order m can be substituted in an arbitrary polynomial; in particular,

$$\langle A\rangle_0 = E, \qquad \langle A\rangle_n = A(A-E)\cdots(A-(n-1)E), \quad n = 1, 2, \dots,$$

where E is the identity matrix of order m. If all the entries of A belong to an algebraic extension $\mathbb{K}$ of $\mathbb{Q}$, then the following definition looks very natural.

Definition 2′. A *matrix* A with entries in $\mathbb{K}$ *has the property of the cancellation of factorials with constant* $\Psi \geqslant 1$ if there exist positive integers $\{\psi_k\}_{k\in\mathbb{N}}$ such that the matrices

$$\psi_k \frac{\langle A\rangle_n}{n!}, \qquad n = 0, 1, \dots, k, \quad k \in \mathbb{N},$$

have entries in $\mathbb{Z}_{\mathbb{K}}$ and

$$\varlimsup_{k\to\infty} \psi_k^{1/k} \leqslant \Psi.$$

Lemma 5. *Let* A *be a matrix with the property of the cancellation of factorials with constant* Ψ. *Then the matrix* TAT^{-1}, *where* T *is an arbitrary non-singular matrix with algebraic entries, has the same property.*

Proof. This can be established on the basis of the elementary identity

$$(TAT^{-1})^n = TA^nT^{-1}, \qquad n = 0, 1, 2, \dots,$$

which shows, in particular, that

$$\frac{\langle TAT^{-1}\rangle_n}{n!} = T\frac{\langle A\rangle_n}{n!}T^{-1}, \qquad n = 0, 1, 2, \dots .$$

If t_1 and t_2 are the least common denominators of the entries of T and T^{-1}, respectively, and $\{\psi_k\}_{k\in\mathbb{N}}$ is the sequence in Definition 2′ corresponding to the matrix A, then for the sequence corresponding to TAT^{-1} we can take $\{t_1t_2\psi_k\}_{k\in\mathbb{N}}$. It remains to observe that

$$\varlimsup_{k\to\infty} (t_1t_2\psi_k)^{1/k} = \varlimsup_{k\to\infty} \psi_k^{1/k}.$$

The proof is complete.

It follows from Lemma 5 that it suffices to establish cancellation of factorials and calculate the corresponding constant for matrices in Jordan normal form.

Lemma 6. *Let A be a matrix formed from Jordan blocks $A_1, \dots, A_s$ at the main diagonal corresponding to (not necessarily distinct) eigenvalues $\lambda_1, \dots, \lambda_s$, respectively. Then the matrix $\Delta_n(A) = \langle A \rangle_n / n!$ contains only blocks $\Delta_n(A_1), \dots, \Delta_n(A_s)$ that lie at the main diagonal and have the following form:*

$$\Delta_n(A_l) = \begin{pmatrix} \Delta_n(\lambda_j) & \frac{1}{1!}\Delta_n^{(1)}(\lambda_j) & \frac{1}{2!}\Delta_n^{(2)}(\lambda_j) & \frac{1}{3!}\Delta_n^{(3)}(\lambda_j) & \dots \\ 0 & \Delta_n(\lambda_j) & \frac{1}{1!}\Delta_n^{(1)}(\lambda_j) & \frac{1}{2!}\Delta_n^{(2)}(\lambda_j) & \dots \\ 0 & 0 & \Delta_n(\lambda_j) & \frac{1}{1!}\Delta_n^{(1)}(\lambda_j) & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \Delta_n(\lambda_j) \end{pmatrix},$$
$$l = 1, \dots, s.$$

Proof. This is a well-known result. Moreover, it remains valid after the replacement of the polynomial $\Delta_n(\,\cdot\,)$ by an arbitrary analytic function in a disc $|z| < C$ containing the eigenvalues $\lambda_1, \dots, \lambda_s$ (see, for instance, [13], Chapter 5, § 1, Example 2).

Remark 1. As follows from Lemma 6, the main diagonal of the matrix $\Delta_n(A)$ contains the quantities $\Delta_n(\lambda_1), \dots, \Delta_n(\lambda_s)$; in accordance with the remark to Lemma 2 a discussion of the cancellation of factorials for the matrix A makes sense only for rational $\lambda_1, \dots, \lambda_s$. Hence we content ourselves in what follows with *rational square matrices* that have rational entries and rational eigenvalues. By the *denominator* $\operatorname{den} A$ of a rational matrix A we shall mean the least common denominator of its eigenvalues. At the same time the entries of the matrix bA are not necessarily integers; one example is the matrix

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

with denominator 1 (its eigenvalues are 0 and 1).

Remark 2. For each eigenvalue λ_l of a rational matrix A we consider the positive integer r_l that is the maximum order of a Jordan block corresponding to the eigenvalue λ_l in the normal form of A. We point out that

$$(\lambda - \lambda_1)^{r_1} (\lambda - \lambda_2)^{r_2} \cdots (\lambda - \lambda_s)^{r_s}, \qquad \lambda_j \leqslant \lambda_l, \quad j \neq l, \tag{17}$$

is called the *minimal polynomial* of A; it divides each polynomial $P(\lambda)$ such that $P(A) = 0$. In particular, by the Cayley–Hamilton theorem the minimal polynomial divides the characteristic polynomial $\det(A - \lambda E)$ of A. In addition, if ${}^tA$ is the transpose of A, then the minimal (and the characteristic) polynomials of ${}^tA$ and A are the same.

Lemma 7. *Let $\lambda \in \mathbb{Q}$, let $b = \operatorname{den} \lambda \in \mathbb{N}$, and let $r \in \mathbb{N}$. Then the least common denominator ψ_k, $k \in \mathbb{N}$, of the quantities*

$$\frac{\Delta_n^{(j)}(\lambda)}{j!}, \qquad j = 0, 1, \dots, r-1, \quad n = 0, 1, \dots, k,$$

divides

$$b^k d_k^{r-1} \prod_{p|b} p^{\tau_p(k)},$$

where d_k *is the least common multiple of* $1, 2, \ldots, k$ *and* $\tau_p(k)$ *is the power of the prime* p *in the factorization of* $k!$ (see (4)).

Proof. Setting in the theorem in [14] $L = H = k$, $Q = b$, $x = -b\lambda$, $M = r - 1$, and $\Lambda = 1$ we arrive at the required result.

One immediate consequence of Lemma 7 is as follows.

Lemma 8. *For* $\lambda_1, \ldots, \lambda_s \in \mathbb{Q}$ *let* $b = \mathrm{den}(\lambda_1, \ldots, \lambda_s)$ *be their least common denominator, let* $r_1, \ldots, r_s \in \mathbb{N}$, *and let* $r = \max_l\{r_l\}$. *Then the least common denominator* ψ_k, $k \in \mathbb{N}$, *of the quantities*

$$\frac{\Delta_n^{(j)}(\lambda_l)}{j!}, \qquad j = 0, 1, \ldots, r_l - 1, \quad l = 1, \ldots, s, \quad n = 0, 1, \ldots, k,$$

divides

$$b^k d_k^{r-1} \prod_{p|b} p^{\tau_p(k)},$$

where d_k *is the least common multiple of* $1, 2, \ldots, k$ *and* $\tau_p(k)$ *is the power of the prime* p *in* $k!$.

Combining Lemmas 5, 6, and 8 we obtain the following result.

Lemma 9. *Let* (17) *be the minimal polynomial of a rational matrix* A; *let* $b = \mathrm{den}\, A$; *let* $r = \max_l\{r_l\}$; *and let* t_1 *and* t_2 *be the least common denominators of the entries of the matrices* T *and* T^{-1}, *respectively, where* T *is the matrix of the transition from* A *to its Jordan normal form. Then the least common denominator* ψ_k, $k \in \mathbb{N}$, *of the entries of the matrices*

$$\Delta_n(A), \qquad n = 0, 1, \ldots, k,$$

divides

$$t_1 t_2 b^k d_k^{r-1} \prod_{p|b} p^{\tau_p(k)},$$

where d_k *is the least common multiple of* $1, 2, \ldots, k$ *and* $\tau_p(k)$ *is the power of the prime* p *in* $k!$.

Taking account of the limit relations

$$\overline{\lim_{k\to\infty}}\, d_k^{1/k} = e, \qquad \lim_{k\to\infty} \biggl(\prod_{p|b} p^{\tau_p(k)}\biggr)^{1/k} = e^{\chi(b)}, \tag{18}$$

and Remark 2 to Lemma 6 we arrive at the following final result.

Theorem 3. *Let*

$$P(\lambda) = (\lambda - \lambda_1)^{r_1}(\lambda - \lambda_2)^{r_2} \cdots (\lambda - \lambda_s)^{r_s}, \quad \lambda_1, \dots, \lambda_s \in \mathbb{Q}, \ \lambda_j \neq \lambda_l, \ j \neq l, \tag{19}$$

be a polynomial annihilating a matrix A (for example, $P(\lambda)$ can be its minimal or its characteristic polynomial); let b be the least common denominator of $\lambda_1, \dots, \lambda_s$ (the denominator of A); and let $r = \max_l\{r_l\}$ be the maximum multiplicity of the roots of the polynomial (19)*. Then the matrix A has the property of the cancellation of factorials with constant $be^{\chi(b)+r-1}$, where the function $\chi(\,\cdot\,)$ is defined by formula* (5).

5. Cancellation of factorials for systems of differential equations of Fuchs type. We now return to the question discussed in § 2.

Consider a system (8) of linear differential equations of Fuchs type and the corresponding systems (10) for the derivatives of orders $n = 1, 2, \dots$; we select the polynomial $T(z) = (z - \gamma_1) \cdots (z - \gamma_s)$ in accordance with (9). We complete the matrices of inhomogeneous systems to square matrices by adding lines of zeros:

$$Q(z) = \begin{pmatrix} 0 & 0 & \dots & 0 \\ Q_{10}(z) & Q_{11}(z) & \dots & Q_{1m}(z) \\ Q_{20}(z) & Q_{21}(z) & \dots & Q_{2m}(z) \\ \dots & \dots & \dots & \dots \\ Q_{m0}(z) & Q_{m1}(z) & \dots & Q_{mm}(z) \end{pmatrix}, \tag{20}$$

$$Q^{[n]}(z) = \begin{pmatrix} 0 & 0 & \dots & 0 \\ Q^{[n]}_{10}(z) & Q^{[n]}_{11}(z) & \dots & Q^{[n]}_{1m}(z) \\ Q^{[n]}_{20}(z) & Q^{[n]}_{21}(z) & \dots & Q^{[n]}_{2m}(z) \\ \dots & \dots & \dots & \dots \\ Q^{[n]}_{m0}(z) & Q^{[n]}_{m1}(z) & \dots & Q^{[n]}_{mm}(z) \end{pmatrix}, \qquad n = 1, 2, \dots,$$

and write the recurrence relations (11) in matrix form:

$$Q^{[n]}(z) = \frac{d}{dz} Q^{[n-1]}(z) + Q^{[n-1]}(z) Q(z), \qquad n = 1, 2, \dots,$$

so that

$$\begin{aligned} {}^t\!Q^{[n]}(z) &= \frac{d}{dz} {}^t\!Q^{[n-1]}(z) + {}^t\!Q(z) {}^t\!Q^{[n-1]}(z) = \left(\frac{d}{dz} + {}^t\!Q(z)\right) {}^t\!Q^{[n-1]}(z) \\ &= \left(\frac{d}{dz} + {}^t\!Q(z)\right)^n E, \qquad n = 1, 2, \dots, \end{aligned} \tag{21}$$

where E is the identity matrix of dimension $m + 1$.

Using the expansion (12) we introduce the differential operator

$$D = \frac{d}{dz} + \frac{1}{z - \gamma_1} {}^t\!A_1 + \dots + \frac{1}{z - \gamma_s} {}^t\!A_s, \tag{22}$$

where $A_1, \dots, A_s$ are rational matrices. We now define the cancellation of factorials for the operator (22) by replacing the ring $\mathbb{Z}_{\mathbb{K}}[z]$ by $(\mathbb{Z}_{\mathbb{K}}[z])^{(m+1)\times(m+1)}$ in Definition 1′.

Lemma 10. *If the operator* (22) *has the property of the cancellation of factorials with constant* Ψ, *then the system of differential equations* (8) *also has the property of the cancellation of factorials with constant* Ψ.

Proof. In view of relations (21) we have

$$ {}^{t}Q^{[n]}(z) = D^n E, \qquad n = 1, 2, \dots . \tag{23}$$

If $\{\psi_k\}_{k\in\mathbb{N}}$ is the sequence from the definition of the cancellation of factorials corresponding to the operator D, then it follows from (23) that the matrices

$$\psi_k \frac{T^n(z)\,{}^{t}Q^{[n]}(z)}{n!}, \qquad n = 0, 1, \dots, k, \quad k \in \mathbb{N},$$

have integer entries. This yields the required result.

Unfortunately, Lemma 4 cannot be applied to the differential operator (22) in the case of *arbitrary* matrices $A_1, \dots, A_s$: the identities

$$\frac{T^n(z)D^n}{n!} = \sum_{\substack{n_0, n_1, \dots, n_s \geqslant 0 \\ n_0 + n_1 + \dots + n_s = n}} (z-\gamma_1)^{n-n_1} \cdots (z-\gamma_s)^{n-n_s} \times \frac{\langle {}^{t}A_1\rangle_{n_1}}{n_1!} \cdots \frac{\langle {}^{t}A_s\rangle_{n_s}}{n_s!} \cdot \frac{1}{n_0!}\frac{d^{n_0}}{dz^{n_0}}, \qquad n \in \mathbb{N}, \tag{24}$$

hold only for *commuting* $A_1, \dots, A_s$.

Taking account of (24), Lemma 9, and the estimates (18) we obtain the following result.

Theorem 4. *Let* $s = 1$ *or, for* $s \geqslant 2$, *let* $A_1, \dots, A_s$ *be pairwise commuting matrices; let* $\lambda_1, \dots, \lambda_p \in \mathbb{Q}$ *be the eigenvalues of the rational matrices* $A_1, \dots, A_s$; *let* $b = \operatorname{den}(\lambda_1, \dots, \lambda_p)$; *and let* $r_{jl} \geqslant 0$, $j = 1, \dots, p$, $l = 1, \dots, s$, *be the multiplicity of the eigenvalue* λ_j *in the minimal polynomial of the matrix* A_l; *let* $r = \max_{j,l}\{r_{jl}\}$ *be the maximum multiplicity of the eigenvalues. Then the operator* (22) *has the property of the cancellation of factorials with constant* $be^{\chi(b)+r-1}$, *where the function* $\chi(\,\cdot\,)$ *is defined by* (5).

In accordance with Lemma 10, Theorem 4 yields the following result.

Theorem 5. *Assume that the matrix* (20) *of a system of differential equations* (8) *has the form* (12) *with commuting rational matrices* $A_1, \dots, A_s$. *Let* b *be the least common denominator of the eigenvalues of the matrices* $A_1, \dots, A_s$, *and* r *the maximum multiplicity of these eigenvalues in the minimal polynomials of these rational matrices. Then the system* (8) *has the property of the cancellation of factorials with constant* $be^{\chi(b)+r-1}$.

Remark. In the case of rational, but not commuting matrices $A_1, \dots, A_s$ the question of the constant of the cancellation of factorials for the differential operator (22) is still open. For a generalization of identities (16) to the operator

$$D = \frac{d}{dz} + \frac{A_1}{z-\gamma_1} + \dots + \frac{A_s}{z-\gamma_s} \tag{25}$$

we define matrices $\langle A_1,\dots,A_s\rangle_{\boldsymbol{n}}$, $\boldsymbol{n}=(n_1,\dots,n_s)$, by induction, setting

$$\langle A_1,A_2,\dots,A_s\rangle_{n_1,n_2,\dots,n_s} = \begin{cases} 0 \quad \text{if } \boldsymbol{n}\notin(\mathbb{Z}_+)^s, \\ E \quad \text{if } \boldsymbol{n}=\boldsymbol{0}, \\ (A_1-n_1+1)\langle A_1,A_2,\dots,A_s\rangle_{n_1-1,n_2,\dots,n_s} \\ \qquad +(A_2-n_2+1)\langle A_1,A_2,\dots,A_s\rangle_{n_1,n_2-1,\dots,n_s}+\cdots \\ \qquad +(A_s-n_s+1)\langle A_1,A_2,\dots,A_s\rangle_{n_1,n_2,\dots,n_s-1} \quad \text{if } \boldsymbol{n}\in(\mathbb{Z}_+)^s. \end{cases}$$

If the matrices $A_1,\dots,A_s$ commute, then

$$\langle A_1,\dots,A_s\rangle_{n_1,\dots,n_s}=\frac{(n_1+\dots+n_s)!}{n_1!\cdots n_s!}\cdot\langle A_1\rangle_{n_1}\cdots\langle A_s\rangle_{n_s}, \qquad \boldsymbol{n}=(n_1,\dots,n_s)\in(\mathbb{Z}_+)^s. \tag{26}$$

Using induction on k it is easy to prove the identities

$$\langle A_1+\dots+A_s\rangle_k=\sum_{\substack{\boldsymbol{n}\in(\mathbb{Z}_+)^s\\ |\boldsymbol{n}|=k}}\langle A_1,\dots,A_s\rangle_{\boldsymbol{n}}, \qquad k=0,1,2,\dots,$$

where $|\boldsymbol{n}|=n_1+\dots+n_s$, and

$$D^k=\sum_{l=0}^{k}\binom{k}{l}\Biggl(\sum_{\substack{\boldsymbol{n}\in(\mathbb{Z}_+)^s\\ |\boldsymbol{n}|=k-l}}\frac{\langle A_1,\dots,A_s\rangle_{\boldsymbol{n}}}{(z-\gamma_1)^{n_1}\cdots(z-\gamma_s)^{n_s}}\Biggr)\frac{d^l}{dz^l}, \qquad k=0,1,2,\dots,$$

for the differential operator (25). In particular,

$$D^kE=\sum_{\substack{\boldsymbol{n}\in(\mathbb{Z}_+)^s\\ |\boldsymbol{n}|=k}}\frac{\langle A_1,\dots,A_s\rangle_{\boldsymbol{n}}}{(z-\gamma_1)^{n_1}\cdots(z-\gamma_s)^{n_s}}, \qquad k=0,1,2,\dots. \tag{27}$$

6. Application to generalized hypergeometric functions. A generalized hypergeometric function

$$f(z)=F\left(\begin{matrix}\alpha_1,\dots,\alpha_m\\ \beta_1+1,\dots,\beta_m+1\end{matrix}\,\middle|\, z\right)=\sum_{n=0}^{\infty}\frac{\langle-\alpha_1\rangle_n\cdots\langle-\alpha_m\rangle_n}{\langle-\beta_1-1\rangle_n\cdots\langle-\beta_m-1\rangle_n}z^n, \qquad \beta_1,\dots,\beta_m\notin\{-1,-2,\dots\}, \tag{28}$$

satisfies the linear differential equations ([15], Chapter 5, § 1, Lemma 1 with $t=1$)

$$\bigl((\delta+\beta_1)\cdots(\delta+\beta_m)-z(\delta+\alpha_1)\cdots(\delta+\alpha_m)\bigr)y=\beta_1\cdots\beta_m, \qquad \delta=z\frac{d}{dz},$$

of order m.

For the functions

$$f_1(z) = f(z), \quad f_2(z) = \delta f_1(z), \quad \dots, \quad f_m(z) = \delta f_{m-1}(z) \tag{29}$$

we obtain the system of linear differential equations

$$\begin{gathered}\frac{d}{dz} y_l = \frac{1}{z} y_{l+1}, \qquad l = 1, \dots, m-1, \\ \frac{d}{dz} y_m = \frac{\sigma_1(\boldsymbol{\beta}) - z\sigma_1(\boldsymbol{\alpha})}{z(z-1)} y_m + \frac{\sigma_2(\boldsymbol{\beta}) - z\sigma_2(\boldsymbol{\alpha})}{z(z-1)} y_{m-1} + \cdots \\ + \frac{\sigma_m(\boldsymbol{\beta}) - z\sigma_m(\boldsymbol{\alpha})}{z(z-1)} y_1 - \frac{\sigma_m(\boldsymbol{\beta})}{z(z-1)},\end{gathered} \tag{30}$$

where $\sigma_l(\,\cdot\,)$, $l = 1, \dots, m$, are the Viète symmetric polynomials of degree l, that is,

$$\begin{gathered}(z+\alpha_1)(z+\alpha_2)\cdots(z+\alpha_m) = z^m + \sigma_1(\boldsymbol{\alpha}) z^{m-1} + \cdots + \sigma_{m-1}(\boldsymbol{\alpha}) z + \sigma_m(\boldsymbol{\alpha}), \\ (z+\beta_1)(z+\beta_2)\cdots(z+\beta_m) = z^m + \sigma_1(\boldsymbol{\beta}) z^{m-1} + \cdots + \sigma_{m-1}(\boldsymbol{\beta}) z + \sigma_m(\boldsymbol{\beta}).\end{gathered} \tag{31}$$

Taking account of the equality

$$\begin{gathered}\frac{\sigma_l(\boldsymbol{\beta}) - z\sigma_l(\boldsymbol{\alpha})}{z(z-1)} = \frac{\sigma_l(\boldsymbol{\beta}) - \sigma_l(\boldsymbol{\alpha})}{z-1} - \frac{\sigma_l(\boldsymbol{\beta})}{z}, \qquad l = 1, \dots, m, \\ \frac{1}{z(z-1)} = \frac{1}{z-1} - \frac{1}{z}\end{gathered}$$

we can write the system (30) in the matrix form:

$$\begin{aligned}\frac{d}{dz}\begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_{m-1} \\ y_m \end{pmatrix} &= \frac{1}{z}\begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ \sigma_m(\boldsymbol{\beta}) \end{pmatrix} + \frac{1}{z-1}\begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ -\sigma_m(\boldsymbol{\beta}) \end{pmatrix} \\ &+ \frac{1}{z}\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 \\ -\sigma_m(\boldsymbol{\beta}) & -\sigma_{m-1}(\boldsymbol{\beta}) & \dots & -\sigma_2(\boldsymbol{\beta}) & -\sigma_1(\boldsymbol{\beta}) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_{m-1} \\ y_m \end{pmatrix} \\ &+ \frac{1}{z-1}\begin{pmatrix} 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 \\ \sigma_m(\boldsymbol{\beta}) - \sigma_m(\boldsymbol{\alpha}) & \dots & \sigma_2(\boldsymbol{\beta}) - \sigma_2(\boldsymbol{\alpha}) & \sigma_1(\boldsymbol{\beta}) - \sigma_1(\boldsymbol{\alpha}) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_{m-1} \\ y_m \end{pmatrix}.\end{aligned} \tag{32}$$

Theorem 5 cannot be applied to (32) because the matrices corresponding to the regular singularities $z = 0$ and $z = 1$ do not commute. However, we are not aiming at the *cancellation of factorials* for the system (32). The explicit calculation of the constant of the cancellation of factorials for a hypergeometric differential equation lost its urgency after [10]: the use of Padé approximants of the second kind (in place of the first) as approximating function forms allows one to cancel factorials without great effort (see [10] and [16]). Our aim is to calculate the constant corresponding to the cancellation of factorials for the system of linear differential equations adjoint to the homogeneous part of (30); this will enable us to use the main result of [17] and find estimates of the measure of irrationality of the values of the hypergeometric function (28) (not involving its successive derivatives) at rational points.

In the general case of (8) the adjoint system is obtained by transposition and changing the sign of the matrix $\big(Q_{lj}(z)\big)_{l,j=1,\dots,m}$; in our case it has the following form:

$$\frac{d}{dz}\begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_{m-1} \\ y_m \end{pmatrix} = \Big(\frac{1}{z}A_1 + \frac{1}{z-1}A_2\Big)\begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_{m-1} \\ y_m \end{pmatrix}, \tag{33}$$

where

$$A_1 = \begin{pmatrix} 0 & 0 & \dots & 0 & \sigma_m(\boldsymbol\beta) \\ -1 & 0 & \dots & 0 & \sigma_{m-1}(\boldsymbol\beta) \\ 0 & -1 & \dots & 0 & \sigma_{m-2}(\boldsymbol\beta) \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & -1 & 0 & \sigma_2(\boldsymbol\beta) \\ 0 & \dots & 0 & -1 & \sigma_1(\boldsymbol\beta) \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & \dots & 0 & \sigma_m(\boldsymbol\alpha)-\sigma_m(\boldsymbol\beta) \\ 0 & \dots & 0 & \sigma_{m-1}(\boldsymbol\alpha)-\sigma_{m-1}(\boldsymbol\beta) \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \sigma_2(\boldsymbol\alpha)-\sigma_2(\boldsymbol\beta) \\ 0 & \dots & 0 & \sigma_1(\boldsymbol\alpha)-\sigma_1(\boldsymbol\beta) \end{pmatrix}.$$

The first $m-1$ columns of A_2 are equal to zero, therefore the rank of the matrix is 1. Hence the eigenvalue $\lambda=0$, which has multiplicity $m-1$ in A_2, enters the minimal polynomial with multiplicity 1. The other eigenvalue is equal to the trace of A_2 and is $\gamma=\sigma_1(\boldsymbol\alpha)-\sigma_1(\boldsymbol\beta)=\alpha_1+\dots+\alpha_m-\beta_1-\dots-\beta_m$.

Up to a sign the matrix A_1 is a *Frobenius block* (see [13], Chapter 6, § 6); its characteristic polynomial is

$$\begin{aligned} \det(A_1-\lambda E) &= (-1)^m(\lambda^m-\sigma_1\lambda^{m-1}+\sigma_2\lambda^{m-2}+\dots+(-1)^m\sigma_m) \\ &= (-1)^m(\lambda-\beta_1)(\lambda-\beta_2)\cdots(\lambda-\beta_m). \end{aligned}$$

Hence the eigenvalues of A_1 are $\beta_1,\dots,\beta_m$.

Lemma 11. *Assume that the parameters $\beta_1,\dots,\beta_m$ are pairwise distinct and $\gamma\ne 0$; let b be the least common denominator of $\gamma,\beta_1,\dots,\beta_m$. Then the system* (33) *has the property of the cancellation of factorials with constant $be^{\chi(b)+2}$, where the function $\chi(\,\cdot\,)$ is defined by* (5).

Proof. The eigenvalue β_j of the matrix A_1 corresponds to the eigenvector

$$\begin{pmatrix} t_{1j} \\ t_{2j} \\ \dots \\ t_{mj} \end{pmatrix} = \begin{pmatrix} \sigma_{m-1}(\beta_1,\dots,\beta_{j-1},\beta_{j+1},\dots,\beta_m) \\ \sigma_{m-2}(\beta_1,\dots,\beta_{j-1},\beta_{j+1},\dots,\beta_m) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \sigma_1(\beta_1,\dots,\beta_{j-1},\beta_{j+1},\dots,\beta_m) \\ \sigma_0(\beta_1,\dots,\beta_{j-1},\beta_{j+1},\dots,\beta_m) \end{pmatrix}, \qquad j=1,\dots,m,$$

where $\sigma_l(\,\cdot\,)$ is the coefficient of z^{m-1-l} in the polynomial

$$(z+\beta_1)\cdots(z+\beta_{j-1})(z+\beta_{j+1})\cdots(z+\beta_m).$$

For the transformation matrix $T=(t_{lj})_{l,j=1,\dots,m}$ we set $\widetilde{T}=T^{-1}=(\tilde{t}_{lj})_{l,j=1,\dots,m}$. Then

$$\tilde{t}_{lj}=(-1)^{m+j}\beta_l^{j-1}\cdot\prod_{\substack{k=1\\k\neq l}}^{m}\frac{1}{\beta_l-\beta_k}, \qquad l,j=1,\dots,m,$$

$\widetilde{A}_1=T^{-1}A_1T$ is a diagonal matrix: $\widetilde{A}_1=\operatorname{diag}(\beta_1,\dots,\beta_m)$, and $\widetilde{A}_2=T^{-1}A_2T$ has the following form:

$$\widetilde{A}_2=\begin{pmatrix} a_1 \\ \dots \\ a_m \end{pmatrix}(1\ \dots\ 1), \qquad a_j=-(\beta_j-\alpha_j)\cdot\prod_{\substack{k=1\\k\neq j}}^{m}\frac{\beta_j-\alpha_k}{\beta_j-\beta_k}, \quad j=1,\dots,m.$$

We now write down the matrices $Q^{[n]}(z)$, $n=0,1,2,\dots$, from Definition 3 corresponding to the system of differential equations (33). By (23) and (27) we obtain

$$\begin{aligned} {}^tQ^{[n]}(z) &= \left(\frac{d}{dz}+\frac{1}{z}{}^tA_1+\frac{1}{z-1}{}^tA_2\right)^n E \\ &= \sum_{\substack{n_1,n_2\geqslant 0\\ n_1+n_2=n}} \frac{\langle {}^tA_1,{}^tA_2\rangle_{n_1,n_2}}{z^{n_1}(z-1)^{n_2}}, \qquad n=0,1,2,\dots, \end{aligned} \tag{34}$$

where

$$\langle {}^tA_1,{}^tA_2\rangle_{n_1,n_2}=\begin{cases} 0 & \text{if } n_1<0 \text{ or } n_2<0, \\ E & \text{if } n_1=n_2=0, \\ ({}^tA_1-n_1+1)\langle {}^tA_1,{}^tA_2\rangle_{n_1-1,n_2} & \\ \quad +({}^tA_2-n_2+1)\langle {}^tA_1,{}^tA_2\rangle_{n_1,n_2-1} & \text{otherwise} \end{cases}$$

(see the remark to Theorem 5). We also point out that the expansions into sums of partial fractions

$$\frac{1}{z^{n_1+1}(1-z)^{n_2+1}}=\sum_{k=0}^{n_1}\binom{n_1+n_2-k}{n_2}\frac{1}{z^{k+1}}+\sum_{k=0}^{n_2}\binom{n_1+n_2-k}{n_1}\frac{1}{(1-z)^{k+1}}, \\ n_1,n_2=0,1,2,\dots, \tag{35}$$

and rearrangements of terms in (34) do not allow one to calculate the constant of the cancellation of factorials for the system (33) because the matrices ${}^tA_1, {}^tA_2$ do not commute.

Setting

$$B_1 = {}^t\widetilde{A}_1 = \operatorname{diag}(\beta_1, \dots, \beta_m), \qquad B_2 = {}^t\widetilde{A}_2 = \begin{pmatrix} 1 \\ \dots \\ 1 \end{pmatrix} (a_1 \ \dots \ a_m),$$

from (34) we obtain

$${}^t\!\big(T^{-1}Q^{[n]}(z)T\big) = \sum_{\substack{n_1,n_2\geqslant 0\\ n_1+n_2=n}} \frac{\langle B_1, B_2\rangle_{n_1,n_2}}{z^{n_1}(z-1)^{n_2}}, \qquad n = 0,1,2,\dots. \tag{36}$$

We now fix a pair of non-negative integers n_1, n_2. If $n_2 = 0$, then by Lemma 9 the denominator of the matrix

$$\frac{\langle B_1, B_2\rangle_{n_1,n_2}}{(n_1+n_2)!} = \frac{\langle B_1\rangle_{n_1}}{n_1!}$$

divides $b_0^k \prod_{p\mid b_0} p^{\tau_p(k)}$ for each $k \geqslant n_1 = n_1 + n_2$; here $b_0 = \operatorname{den}(\beta_1, \dots, \beta_m)$.

Next, assume that $n_2 > 0$. The recursive relations for $\langle B_1, B_2\rangle_{n_1,n_2}$ (and formula (26) in the case of commuting matrices) show that the matrix $\langle B_1, B_2\rangle_{n_1,n_2}$ is a sum of $N = (n_1+n_2)!/(n_1!\,n_2!)$ terms, each coinciding with $\langle B_1\rangle_{n_1}\langle B_2\rangle_{n_2}$ up to the order of the factors:

$$\langle B_1, B_2\rangle_{n_1,n_2} = \sum_{r=1}^{N} B^{(r)}. \tag{37}$$

We point out straight away that if at least one of the square matrices X_1 and X_2 is diagonal, then the main diagonals of X_1X_2 and X_2X_1 are the same. Since the matrices $B_1 - lE$, $l = 0, 1, \dots, n_1 - 1$, are diagonal, the main diagonal of each term in (37) is equal to the main diagonal of $\langle B_1\rangle_{n_1}\langle B_2\rangle_{n_2}$. The matrix B_2 satisfies the relation

$$B_2^2 = (a_1 + \dots + a_m)B_2 = \operatorname{Tr} B_2 \cdot B_2 = \gamma B_2,$$

therefore $\langle B_2\rangle_{n_2} = \langle\gamma - 1\rangle_{n_2-1}B_2$ and

$$\langle B_1\rangle_{n_1}\langle B_2\rangle_{n_2} = \langle\gamma-1\rangle_{n_2-1}\cdot \begin{pmatrix} a_1\langle\beta_1\rangle_{n_1} & a_2\langle\beta_1\rangle_{n_1} & \dots & a_m\langle\beta_1\rangle_{n_1} \\ a_1\langle\beta_2\rangle_{n_1} & a_2\langle\beta_2\rangle_{n_1} & \dots & a_m\langle\beta_2\rangle_{n_1} \\ \dots & \dots & \dots & \dots \\ a_1\langle\beta_m\rangle_{n_1} & a_2\langle\beta_m\rangle_{n_1} & \dots & a_m\langle\beta_m\rangle_{n_1} \end{pmatrix}. \tag{38}$$

Consider now an arbitrary term $B = B^{(r)}$, $1 \leqslant r \leqslant N$, of (37). The factor $B_1 - jE$ enters this term to the left of $B_1 - lE$ if and only if $j > l$; the same can be said about the relative order of the factors $B_2 - jE$ and $B_2 - lE$ in B. Distinguishing the first (and, in fact, the only) occurrence of B_2 in B we obtain

$$\begin{aligned} B = X \cdot B_2 \cdot \langle B_1\rangle_s &= X \cdot \begin{pmatrix} 1 \\ \dots \\ 1 \end{pmatrix} (a_1 \ \dots \ a_m) \cdot \operatorname{diag}\bigl(\langle\beta_1\rangle_s, \dots, \langle\beta_m\rangle_s\bigr) \\ &= \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_m \end{pmatrix} \bigl(a_1\langle\beta_1\rangle_s \ a_2\langle\beta_2\rangle_s \ \dots \ a_m\langle\beta_m\rangle_s\bigr) \\ &= \begin{pmatrix} a_1\langle\beta_1\rangle_s x_1 & a_2\langle\beta_2\rangle_s x_1 & \dots & a_m\langle\beta_m\rangle_s x_1 \\ a_1\langle\beta_1\rangle_s x_2 & a_2\langle\beta_2\rangle_s x_2 & \dots & a_m\langle\beta_m\rangle_s x_2 \\ \dots & \dots & \dots & \dots \\ a_1\langle\beta_1\rangle_s x_m & a_2\langle\beta_2\rangle_s x_m & \dots & a_m\langle\beta_m\rangle_s x_m \end{pmatrix}. \end{aligned} \tag{39}$$

A comparison of the main diagonals of (38) and (39) yields

$$x_l = \langle\gamma - 1\rangle_{n_2-1} \cdot \langle\beta_l - s\rangle_{n_1-s}, \qquad l = 1, \dots, m.$$

Thus, the terms in (37) have the following form:

$$B = \langle\gamma - 1\rangle_{n_2-1} \cdot \Bigl(a_j\langle\beta_j\rangle_s\langle\beta_l - s\rangle_{n_1-s}\Bigr)_{l,j=1,\dots,m}$$

with $0 \leqslant s < n_1$. Setting $a = \operatorname{den}(a_1, \dots, a_m)$ and $b = \operatorname{den}(\gamma, \beta_1, \dots, \beta_m)$, denoting by g_k the least common multiple of

$$\frac{k!}{k_0!\,k_1!\,k_2!}, \qquad k_0, k_1, k_2 = 0, 1, 2, \dots, \quad k_0 + k_1 + k_2 = k, \tag{40}$$

and taking account of the condition $\gamma \neq 0$, we obtain by Lemma 9 that the least common denominator of the entries of the matrix

$$\frac{\gamma B}{(n_1+n_2)!} = \frac{s!\,(n_1-s)!\,n_2!}{(n_1+n_2)!} \cdot \frac{\gamma B}{s!\,(n_1-s)!\,n_2!}$$

divides

$$g_k \cdot ab^k \prod_{p\mid b} p^{\tau_p(k)} \tag{41}$$

for each $k \geqslant n_1 + n_2$; since B is an arbitrary term in (37), we conclude that the least common denominator of the entries of

$$\frac{\gamma\langle B_1, B_2\rangle_{n_1,n_2}}{(n_1+n_2)!} = \sum_{r=1}^{N} \frac{\gamma B^{(r)}}{(n_1+n_2)!}$$

also divides (41) for each $k \geqslant n_1 + n_2$.

The degree of a prime p in the (prime) factorization of each integer (40) is by (4) equal to the quantity

$$\tau_p(k) - \tau_p(k_0) - \tau_p(k_1) - \tau_p(k_2) = \sum_{m=1}^{\infty}\left(\left\lfloor\frac{k}{p^m}\right\rfloor - \left\lfloor\frac{k_0}{p^m}\right\rfloor - \left\lfloor\frac{k_1}{p^m}\right\rfloor - \left\lfloor\frac{k_2}{p^m}\right\rfloor\right), \tag{42}$$
$$k_0, k_1, k_2 = 0, 1, 2, \dots, \qquad k_0 + k_1 + k_2 = k.$$

Summation in (42) proceeds only for $m \leqslant \lfloor \log k/\log p\rfloor$; in addition,

$$\lfloor \xi_0 + \xi_1 + \xi_2 \rfloor - \lfloor \xi_1 \rfloor - \lfloor \xi_2 \rfloor - \lfloor \xi_3 \rfloor \leqslant 2, \qquad \xi_0, \xi_1, \xi_2 \in \mathbb{R}.$$

Hence for all primes $p \leqslant k$ we have

$$\tau_p(k) - \tau_p(k_0) - \tau_p(k_1) - \tau_p(k_2) \leqslant 2\left\lfloor\frac{\log k}{\log p}\right\rfloor \leqslant 2\frac{\log k}{\log p},$$
$$k_0, k_1, k_2 = 0, 1, 2, \dots, \qquad k_0 + k_1 + k_2 = k.$$

These inequalities yield the estimate

$$g_k \leqslant \prod_{p\leqslant k} p^{2\log k/\log p} = e^{2\pi(k)\log k},$$

where $\pi(k)$ is the number of primes not exceeding k, therefore

$$\varlimsup_{k\to\infty} g_k^{1/k} \leqslant e^2. \tag{43}$$

Taking account of this limit relation, identities (36), and Lemma 5 we see that the system of homogeneous differential equations (33) has the property of the cancellation of factorials with constant $be^{\chi(b)+2}$. The proof is complete.

Remark 1. Repeating the above arguments in the case $\gamma = 0$ for each term in (37) we obtain the equality

$$B = (-1)^{n_2-1}(n_2-1)! \cdot \Big(a_j\langle\beta_j\rangle_s\langle\beta_l - s\rangle_{n_1-s}\Big)_{l,j=1,\dots,m}$$

for some s, $0 \leqslant s < n_1$, and therefore the least common denominator of the entries of the matrix

$$\frac{\langle B_1, B_2\rangle_{n_1,n_2}}{(n_1+n_2)!}$$

divides the integer

$$d_k g_k \cdot ab^k \prod_{p|b} p^{\tau_p(k)}$$

for each $k \geqslant n_1 + n_2$, where d_k is the least common multiple of $1, 2, \dots, k$. In view of limit relations (18) and (43) the cancellation of factorials occurs in this case with constant $be^{\chi(b)+3}$.

Remark 2. The proof of Lemma 11 is suitable for the calculation of the (actually, the same) constant of the cancellation of factorials for the original inhomogeneous system (32) in the case of pairwise distinct parameters $\beta_1, \dots, \beta_m$.

The following results are related to the arithmetic and algebraic properties of the functions (29).

Lemma 12. *Let b_1 and b_2 be the denominators of numbers $\alpha, \beta \in \mathbb{Q}\backslash\{-1,-2,\dots\}$; let b be the least common multiple of b_1 and b_2. Also, let $\varphi_k \in \mathbb{N}$, $k \in \mathbb{N}$, be the least common multiple of the rational numbers*

$$\frac{\langle -\alpha\rangle_n}{\langle -\beta\rangle_n}, \qquad n = 0, 1, \dots, k.$$

Then

$$\varlimsup_{k\to\infty} \varphi_k^{1/k} \leqslant \Phi = e^{\rho(b_2)} \frac{b_1}{b},$$

where

$$\rho(b) = \frac{b}{\varphi(b)} \sum_{\substack{1\leqslant n\leqslant b\\ (n,b)=1}} \frac{1}{n}, \quad \varphi(b) = \sum_{\substack{1\leqslant n\leqslant b\\ (n,b)=1}} 1, \qquad b \in \mathbb{N}. \tag{44}$$

Proof. The least common denominators of the numbers

$$\langle -\alpha\rangle_n, \quad n = 0, 1, \dots, k, \quad \text{and} \quad \langle -\beta\rangle_n, \quad n = 0, 1, \dots, k,$$

are b_1^k and b_2^k, respectively. The least common denominator of the numbers

$$\frac{1}{\langle -\beta\rangle_n}, \qquad n = 0, 1, \dots, k,$$

is equal to the least common multiple d_k of the numbers

$$-a + b_2(n-1), \quad n = 1, \dots, k, \qquad a = b_2\beta \in \mathbb{Z}.$$

According to [17], Lemma 3.2 we have the estimate

$$\varlimsup_{k\to\infty} \frac{\log d_k}{k} \leqslant \rho(b_2),$$

where the function $\rho(\,\cdot\,)$ is defined by (44). This completes the proof.

Lemma 13. *Let q_1 and q_2 be the products of the denominators of rational numbers $\alpha_1, \dots, \alpha_m$ and $\beta_1, \dots, \beta_m$, respectively; let b be the least common multiple of q_1 and q_2; let $b_1, \dots, b_m$ be the denominators of $\beta_1, \dots, \beta_m$, respectively. Then the function* (28) *belongs to the class* $\mathbf{G}(1, \Phi)$, *where* $\Phi = e^{\rho(b_1)+\dots+\rho(b_m)} q_1/b$.

Proof. The estimate of the quantity Φ follows from Lemma 12. Since

$$\lim_{n\to\infty} \left|\frac{\langle -\lambda\rangle_n}{n!}\right|^{1/n} = 1, \qquad \lambda \notin \{-1, -2, \dots\},$$

the convergence domain of (28) is the disc $|z| < 1$. The proof is complete.

Lemma 14. *If $f(z) \in \mathbf{G}(C,\Phi)$, then $\delta f(z) \in \mathbf{G}(C,\Phi)$ for $\delta = z\frac{d}{dz}$.*

Proof. This is obvious. If the Taylor series

$$f(z) = \sum_{n=0}^{\infty} f_n z^n$$

of $f(z)$ converges for $|z| < C$, then the series

$$\delta f(z) = \sum_{n=1}^{\infty} n f_n z^n$$

also converges in this disc. If we choose a sequence of positive integers $\{\varphi_k\}_{k\in\mathbb{N}}$ such that

$$\varphi_k f_n \in \mathbb{Z}, \qquad n = 0,1,\dots,k, \quad k \in \mathbb{N},$$

then

$$\varphi_k n f_n \in \mathbb{Z}, \qquad n = 0,1,\dots,k, \quad k \in \mathbb{N}.$$

Thus, $\delta f(z) \in \mathbf{G}(C,\Phi)$, as required.

Corollary. *Let q_1 and q_2 be the products of the denominators of rational numbers $\alpha_1,\dots,\alpha_m$ and $\beta_1,\dots,\beta_m$, respectively; let b be the least common multiple of q_1 and q_2; let $b_1,\dots,b_m$ be the denominators of $\beta_1,\dots,\beta_m$, respectively. Then the family of functions* (29) *belongs to the class $\mathbf{G}(1,\Phi)$ for $\Phi = e^{\rho(b_1)+\dots+\rho(b_m)} q_1/b$.*

Lemma 15. *The Wronskian* (*the determinant of the matrix of a fundamental system of solutions*) *$W(z)$ of a homogeneous linear differential equation of order m*

$$\big((\delta+\beta_1)\cdots(\delta+\beta_m) - z(\delta+\alpha_1)\cdots(\delta+\alpha_m)\big)y = 0, \qquad \delta = z\frac{d}{dz}, \tag{45}$$

satisfies the differential equation

$$\big((\delta+\beta) - z(\delta+\alpha)\big)y = 0, \qquad \alpha = \sigma_1(\boldsymbol{\alpha}), \quad \beta = \sigma_1(\boldsymbol{\beta}), \tag{46}$$

and therefore, for rational α and β, is an algebraic function:

$$W(z) = Cz^{-\beta}(1-z)^{\alpha-\beta}, \qquad C \in \mathbb{C}. \tag{47}$$

Proof. The Wronskian $W(z)$ of differential equation (45) is equal to the Wronskian of the system of linear differential equations

$$\begin{gathered} \frac{d}{dz}y_l = \frac{1}{z}y_{l+1}, \qquad l = 1,\dots,m-1, \\ \frac{d}{dz}y_m = \frac{\sigma_1(\boldsymbol{\beta}) - z\sigma_1(\boldsymbol{\alpha})}{z(z-1)}y_m + \frac{\sigma_2(\boldsymbol{\beta}) - z\sigma_2(\boldsymbol{\alpha})}{z(z-1)}y_{m-1} + \dots + \frac{\sigma_m(\boldsymbol{\beta}) - z\sigma_m(\boldsymbol{\alpha})}{z(z-1)}y_1. \end{gathered} \tag{48}$$

By Liouville's theorem ([18], Chapter 3, § 27.6), it satisfies the equation

$$\frac{d}{dz}y = \operatorname{Tr} Q(z) \cdot y, \tag{49}$$

where $\operatorname{Tr} Q(z) = (\beta - z\alpha)/(z(z-1))$ is the trace of the matrix of the system (48). Equation (49) can be written in the form (46). Its solution (47) can be obtained by direct integration.

We now state several sufficient conditions from [19] on the parameters of the function (28) ensuring that the functions (29) are algebraically independent over the field $\mathbb{C}(z)$:

(1) *linear irreducibility*: $\alpha_l - \beta_j \notin \mathbb{Z}$ for all $l, j = 1, \dots, m$;

(2) *Belyĭ irreducibility* ([19], Chapter 3, Lemma 3.5.3): for each pair of positive integers m_1 and m_2, $m_1 + m_2 = m$, there exist no $u, v \in \mathbb{Q}$ such that either

$$(\alpha_1, \alpha_2, \dots, \alpha_m) \sim \left(\frac{u}{m_1}, \frac{u+1}{m_1}, \dots, \frac{u+m_1-1}{m_1}, \frac{v}{m_2}, \frac{v+1}{m_2}, \dots, \frac{v+m_2-1}{m_2}\right),$$
$$(\beta_1, \beta_2, \dots, \beta_m) \sim \left(\frac{u+v}{m}, \frac{u+v+1}{m}, \dots, \frac{u+v+m-1}{m}\right),$$

or

$$(\alpha_1, \alpha_2, \dots, \alpha_m) \sim \left(\frac{u+v}{m}, \frac{u+v+1}{m}, \dots, \frac{u+v+m-1}{m}\right),$$
$$(\beta_1, \beta_2, \dots, \beta_m) \sim \left(\frac{u}{m_1}, \frac{u+1}{m_1}, \dots, \frac{u+m_1-1}{m_1}, \frac{v}{m_2}, \frac{v+1}{m_2}, \dots, \frac{v+m_2-1}{m_2}\right);$$

(3) *Kummer irreducibility* ([19], Chapter 3, Lemma 3.5.6): the integer m has no divisor $m_0 \geqslant 2$ such that

$$(\alpha_1, \alpha_2, \dots, \alpha_m) \sim \left(\alpha_1 + \frac{1}{m_0}, \alpha_2 + \frac{1}{m_0}, \dots, \alpha_m + \frac{1}{m_0}\right),$$
$$(\beta_1, \beta_2, \dots, \beta_m) \sim \left(\beta_1 + \frac{1}{m_0}, \beta_2 + \frac{1}{m_0}, \dots, \beta_m + \frac{1}{m_0}\right);$$

(4) $2\gamma \notin \mathbb{Z}$, where $\gamma = \alpha_1 + \dots + \alpha_m - \beta_1 - \dots - \beta_m$.

The notation $(\lambda_1, \dots, \lambda_m) \sim (\lambda'_1, \dots, \lambda'_m)$ indicates that for some permutation $\sigma\colon \{1, \dots, m\} \to \{1, \dots, m\}$ for all $l = 1, \dots, m$ we have $\lambda_l - \lambda'_{\sigma(l)} \in \mathbb{Z}$.

Lemma 16. *Let $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m \in \mathbb{Q} \setminus \{-1, -2, \dots\}$ be numbers satisfying conditions* (1)–(4) *above. Then the functions* (29), *where $f(z)$ is defined by the series* (28), *are algebraically independent over the field $\mathbb{C}(z)$.*

Proof. By [19], Chapter 3, Theorem 3.5.8, if conditions (1)–(4) are satisfied, then the Galois group of homogeneous linear differential equation (45) of order m is isomorphic to $\mathrm{SL}_m(\mathbb{C})$. This means that the functions in the fundamental system of solutions of (45) are related by means of only one algebraic relation over the field $\mathbb{C}(z)$: the determinant of this fundamental system is an algebraic system (Lemma 15). Hence if $g(z)$ is an arbitrary non-trivial solution of (45), then the functions $g(z), \delta g(z), \dots, \delta^{m-1} g(z)$ are algebraically independent over $\mathbb{C}(z)$. By Nesterenko's theorem ([20], Theorem 2; see also [15], Chapter 9, § 6, Theorem 2) either the functions (29) are algebraically independent over $\mathbb{C}(z)$ or they all belong to $\mathbb{C}(z)$. The second case is impossible because (28) is not a rational function. The proof is complete.

Theorem 6. *Assume that the parameters $\alpha_1, \dots, \alpha_m$ and $\beta_1, \dots, \beta_m$ of the function* (28) *satisfy conditions* (1)–(4), *let $\beta_1, \dots, \beta_m$ be distinct, let ξ be a rational number, $\xi = a_1/a_2 \ne 0$ with $a_2 = \operatorname{den} \xi \in \mathbb{N}$, and let $\varepsilon < 1/(m+2)$ be an arbitrary positive constant. Let b_0 be the least common denominator of $\gamma, \beta_1, \dots, \beta_m$, let q_1 and q_2 be the products of the denominators of $\alpha_1, \dots, \alpha_m$ and $\beta_1, \dots, \beta_m$, respectively, b the least common multiple of q_1 and q_2, and H the maximum of the absolute values of the coefficients of* (31). *Also let $\Phi = e^{\rho(\operatorname{den} \beta_1) + \dots + \rho(\operatorname{den} \beta_m)} q_1/b$,*

$$C_0 = \left(8 b_0 H e^{\chi(b_0)+3}\right)^{\varepsilon(1-\log \varepsilon)} \Phi^{1+\varepsilon+(2-(m-1)\varepsilon)/(\varepsilon^m (m-1)!)},$$

$$\eta_0 = \frac{(1+\varepsilon)\log a_2 + \log C_0}{(1-(m+2)\varepsilon)\log a_2 - \log C_0 - (2-(m+1)\varepsilon)\log|a_1|},$$

where the functions $\chi(\,\cdot\,)$ and $\rho(\,\cdot\,)$ are defined by (5) *and* (44), *respectively. If for some ξ condition $\eta_0 > 0$ is satisfied, that is, if*

$$a_2^{1-(m+2)\varepsilon} > C_0 |a_1|^{2-(m+1)\varepsilon},$$

then $f(\xi)$ is an irrational number. Moreover, the estimate

$$\left| f(\xi) - \frac{p}{q} \right| > q^{-1-\eta}$$

holds for each $\eta > \eta_0$ and arbitrary $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, $q > q_*(\xi, \varepsilon, \eta)$.

Proof. Let $T_0(z) = bT(z) = bz(z-1)$ be the common denominator of the system of differential equations (32) which hold for the family of functions (29) in the class $\mathbf{G}(1, \Phi)$ (see the corollary to Lemma 14); in addition,

$$T_0(z) Q_{lj}(z) \in \mathbb{Z}[z], \qquad l = 1, \dots, m, \quad j = 0, \dots, m.$$

Then

$$\max\Bigl\{\deg T_0 - 1, \max_{l,j}\{\deg T_0 Q_{lj}\}\Bigr\} = 1, \qquad \max\Bigl\{H(T_0), \max_{l,j}\{H(T_0 Q_{lj})\}\Bigr\} = bH.$$

It follows from condition (4) that $\gamma \ne 0$. By Lemma 11, the cancellation of factorials for the system of differential equations (33) adjoint to the homogeneous part of (32) holds with the constant $\Psi = b_0 e^{\chi(b_0)+2}$ (with the constant $\Psi = b_0 e^{\chi(b_0)+2}/b$ if the polynomial $T(z)$ in Definition 3 is replaced by $T_0(z)$). Lemma 16 yields the algebraic independence of the functions (29) over the field $\mathbb{C}(z)$. Applying now the main theorem and inequalities (0.9) of [17] we arrive at the required result.

7. Cancellation of factorials for homogeneous linear differential equations with constant coefficients. We consider the system of homogeneous linear differential equations

$$\frac{d}{dz}y_l = \sum_{j=1}^{m} A_{lj} y_j, \quad l = 1, \dots, m, \qquad A_{lj} \in \mathbb{C}, \quad l, j = 1, \dots, m, \tag{50}$$

and the associated differential operator

$$\mathcal{A} = [A] = \sum_{l=1}^{m} \sum_{j=1}^{m} A_{lj} y_j \frac{\partial}{\partial y_l}, \qquad A = \begin{pmatrix} A_{11} & \dots & A_{1m} \\ \dots & \dots & \dots \\ A_{m1} & \dots & A_{mm} \end{pmatrix}. \tag{51}$$

Thus, with each matrix A we can associate the linear differential operator $[A]$ by formula (51). We note first of all the property of linearity:

$$[\lambda_1 A_1 + \lambda_2 A_2] = \lambda_1 [A_1] + \lambda_2 [A_2], \qquad \lambda_1, \lambda_2 \in \mathbb{C},$$

for all square matrices A_1 and A_2 of dimension m.

For differential operators $[A]$ and $[B]$ we define the operations of formal multiplication

$$\begin{aligned} [A] \cdot [B] &= \Bigl(\sum_{l=1}^{m} \sum_{j=1}^{m} A_{lj} y_j \frac{\partial}{\partial y_l}\Bigr) \cdot \Bigl(\sum_{i=1}^{m} \sum_{k=1}^{m} B_{ik} y_k \frac{\partial}{\partial y_i}\Bigr) \\ &= \sum_{l=1}^{m} \sum_{i=1}^{m} \Bigl(\sum_{j=1}^{m} A_{lj} y_j\Bigr) \Bigl(\sum_{k=1}^{m} B_{ik} y_k\Bigr) \frac{\partial^2}{\partial y_l \, \partial y_i} = [B] \cdot [A] \end{aligned}$$

and taking the composite

$$\begin{aligned} [B] \circ [A] &= \Bigl(\sum_{i=1}^{m} \sum_{k=1}^{m} B_{ik} y_k \frac{\partial}{\partial y_i}\Bigr) \circ \Bigl(\sum_{l=1}^{m} \sum_{j=1}^{m} A_{lj} y_j \frac{\partial}{\partial y_l}\Bigr) \\ &= \sum_{i=1}^{m} \sum_{k=1}^{m} B_{ik} y_k \sum_{l=1}^{m} A_{li} \frac{\partial}{\partial y_l} + \Bigl(\sum_{i=1}^{m} \sum_{k=1}^{m} B_{ik} y_k \frac{\partial}{\partial y_i}\Bigr) \cdot \Bigl(\sum_{l=1}^{m} \sum_{j=1}^{m} A_{lj} y_j \frac{\partial}{\partial y_l}\Bigr) \\ &= \sum_{l=1}^{m} \sum_{k=1}^{m} \sum_{i=1}^{m} A_{li} B_{ik} y_k \frac{\partial}{\partial y_l} + \sum_{l=1}^{m} \sum_{i=1}^{m} \Bigl(\sum_{j=1}^{m} A_{lj} y_j\Bigr) \Bigl(\sum_{k=1}^{m} B_{ik} y_k\Bigr) \frac{\partial^2}{\partial y_l \, \partial y_i} \\ &= [AB] + [A] \cdot [B]. \end{aligned}$$

Obviously associative, both operations are distributive over addition. Commutativity holds only for formal multiplication, although the order of factors in the composite

$$\mathcal{A}_n = ([A] - n + 1) \circ ([A] - n + 2) \circ \dots \circ ([A] - 1) \circ [A] \tag{52}$$

is of no importance. We point out also the 'Leibniz property' of taking the composite:

$$[B] \circ ([A_1] \cdot [A_2]) = ([B] \circ [A_1]) \cdot [A_2] + [A_1] \cdot ([B] \circ [A_2]).$$

By $[A]^n$ we shall mean the formal product of n operators $[A]$.

Definition 4. We say that the *differential operator* (51) (or the *system of equations* (50)) associated with the matrix A *has the property of the cancellation of factorials with constant* $\Psi \geqslant 1$ if there exists a sequence of positive integers $\{\psi_k\}_{k\in\mathbb{N}}$ such that the operators

$$\psi_k \frac{1}{n!}\mathcal{A}_n, \qquad n = 0, 1, \dots, k, \quad k \in \mathbb{N}, \tag{53}$$

take the ring $\mathbb{Z}[y_1, \dots, y_m]$ into itself and

$$\varlimsup_{k\to\infty} \psi_k^{1/k} \leqslant \Psi.$$

Lemma 17. *The differential operator* (52) *satisfies the identity*

$$\frac{1}{n!}\mathcal{A}_n = \sum_{\substack{s_1,s_2,\dots,s_n\geqslant 0\\ s_1+2s_2+\dots+ns_n=n}} \frac{1}{s_1!}[A]^{s_1} \cdot \frac{1}{s_2!}\left[\frac{\langle A\rangle_2}{2!}\right]^{s_2} \cdots \frac{1}{s_n!}\left[\frac{\langle A\rangle_n}{n!}\right]^{s_n}, \tag{54}$$

where the symbol $\langle\,\cdot\,\rangle_n$ *is defined by* (1).

Proof. We proceed by induction on n. The basis of induction $n = 1$ is obvious. Assume that (54) holds for some n. Then we set

$$U(s_1, s_2, \dots, s_n) = \frac{1}{s_1!}[A]^{s_1} \cdot \frac{1}{s_2!}\left[\frac{\langle A\rangle_2}{2!}\right]^{s_2} \cdots \frac{1}{s_n!}\left[\frac{\langle A\rangle_n}{n!}\right]^{s_n} \tag{55}$$

and write (54) in the following form:

$$\begin{aligned}\frac{1}{n!}\mathcal{A}_n &= \sum_{\substack{s_1,s_2,\dots,s_n\geqslant 0\\ s_1+2s_2+\dots+ns_n=n}} U(s_1, s_2, \dots, s_n)\\ &= \sum_{\substack{s_1,s_2,\dots,s_n,s_{n+1}\geqslant 0\\ s_1+2s_2+\dots+ns_n+(n+1)s_{n+1}=n}} U(s_1, s_2, \dots, s_n, s_{n+1})\end{aligned}$$

(here, obviously, $s_{n+1} = 0$). Since

$$([A] - i) \circ [\langle A\rangle_i] = [A] \cdot [\langle A\rangle_i] + [\langle A\rangle_{i+1}],$$

it follows that

$$\begin{aligned}&([A] - n) \circ U(s_1, s_2, \dots, s_n, s_{n+1})\\ &\qquad = (s_1 + 1)U(s_1 + 1, s_2, \dots, s_n, s_{n+1})\\ &\qquad\quad + 2(s_2 + 1)U(s_1 - 1, s_2 + 1, s_3, \dots, s_n, s_{n+1})\\ &\qquad\quad + 3(s_3 + 1)U(s_1, s_2 - 1, s_3 + 1, s_4, \dots, s_n, s_{n+1}) + \cdots\\ &\qquad\quad + (n + 1)(s_{n+1} + 1)U(s_1, s_2, \dots, s_{n-1}, s_n - 1, s_{n+1} + 1),\\ &s_1, s_2, \dots, s_n, s_{n+1} \geqslant 0, \qquad s_1 + 2s_2 + \dots + ns_n + (n + 1)s_{n+1} = n.\end{aligned}$$

Hence

$$\begin{aligned}\frac{1}{(n+1)!}\mathcal{A}_{n+1} &= \frac{[A]-n}{n+1}\circ\frac{1}{n!}\mathcal{A}_n\\ &= \sum_{\substack{s_1,s_2,\dots,s_n,s_{n+1}\geqslant 0\\ s_1+2s_2+\cdots+ns_n+(n+1)s_{n+1}=n}} \frac{s_1+1}{n+1}U(s_1+1,s_2,\dots,s_n,s_{n+1})\\ &\quad+\frac{2(s_2+1)}{n+1}U(s_1-1,s_2+1,s_3,\dots,s_n,s_{n+1})+\cdots\\ &\quad+\frac{(n+1)(s_{n+1}+1)}{n+1}U(s_1,s_2,\dots,s_{n-1},s_n-1,s_{n+1}+1)\\ &= \sum_{\substack{s_1',s_2',\dots,s_{n+1}'\geqslant 0\\ s_1'+2s_2'+\cdots+(n+1)s_{n+1}'=n+1}} \frac{s_1'+2s_2'+\cdots+(n+1)s_{n+1}'}{n+1}U(s_1',s_2',\dots,s_{n+1}')\\ &= \sum_{\substack{s_1',s_2',\dots,s_{n+1}'\geqslant 0\\ s_1'+2s_2'+\cdots+(n+1)s_{n+1}'=n+1}} U(s_1',s_2',\dots,s_{n+1}'),\end{aligned}$$

which demonstrates (54) for $n+1$. The proof is complete.

Lemma 18. *If B is a matrix with integer entries, then the differential operator $[B]^s/s!$ takes the ring $\mathbb{Z}[y_1,\dots,y_m]$ into itself.*

Proof. We write the differential operator $[B]$ in the following form:

$$[B]=\sum_{l=1}^{m} b_l\frac{\partial}{\partial y_l},\qquad b_l=\sum_{j=1}^{m} B_{lj}y_j\in\mathbb{Z}[y_1,\dots,y_m].$$

Then the differential operator

$$\frac{1}{s!}[B]^s=\frac{1}{s!}\left(\sum_{l=1}^{m} b_l\frac{\partial}{\partial y_l}\right)^s=\sum_{\substack{l_1,\dots,l_m\geqslant 0\\ l_1+\cdots+l_m=s}}\frac{b_1^{l_1}}{l_1!}\frac{\partial^{l_1}}{\partial y_1^{l_1}}\cdots\frac{b_m^{l_m}}{l_m!}\frac{\partial^{l_m}}{\partial y_m^{l_m}}$$

takes the ring $\mathbb{Z}[y_1,\dots,y_m]$ into itself because the operators

$$\frac{1}{l_j!}\frac{\partial^{l_j}}{\partial y_j^{l_j}},\qquad j=1,\dots,m$$

(Example 1) have this property and, in addition,

$$b_j^{l_j}\in\mathbb{Z}[y_1,\dots,y_m],\qquad j=1,\dots,m.$$

The proof is complete.

Lemma 19. *Let $s_1, s_2, \dots, s_k$ be non-negative integers and let p be a prime. Then*

$$s_1\tau_p(1) + s_2\tau_p(2) + \dots + s_k\tau_p(k) \leqslant \tau_p(s_1 + 2s_2 + \dots + ks_k), \tag{56}$$

where $\tau_p(k)$ is the power of p in the factorization of $k!$ (see (4)).

Proof. We point out first of all that for real $\xi_1, \dots, \xi_k$,

$$\lfloor\xi_1\rfloor + \dots + \lfloor\xi_k\rfloor \leqslant \lfloor\xi_1 + \dots + \xi_k\rfloor$$

(see [2], Part 8, Chapter 1, § 1, Problem 7). Hence

$$s_1\lfloor\xi_1\rfloor + s_2\lfloor\xi_2\rfloor + \dots + s_k\lfloor\xi_k\rfloor \leqslant \lfloor s_1\xi_1 + s_2\xi_2 + \dots + s_k\xi_k\rfloor. \tag{57}$$

Substituting in (57) one after another the values

$$\xi_n = \frac{n}{p}, \frac{n}{p^2}, \frac{n}{p^3}, \dots, \qquad n = 1, \dots, k,$$

and adding the resulting inequalities, in view of (4), we obtain (56). The proof is complete.

Theorem 7. *Assume that the differential operator* (51) *is accociated with a rational matrix A,* $\operatorname{den} A = b$, *whose minimal polynomial has no multiple zeros; let t_1 and t_2 be the least common denominators of the entries of T and T^{-1}, respectively, where T is a transformation matrix bringing A to Jordan normal form. Then the operator $[A]$ has the property of the cancellation of factorials with constant $t_1t_2be^{\chi(b)}$.*

Proof. Let k be an arbitrary positive integer. By Lemma 9 the entries of the matrices

$$t_1t_2b^n \prod_{p|b} p^{\tau_p(n)} \cdot \frac{\langle A\rangle_n}{n!}, \qquad n = 0, 1, \dots, k,$$

are integers. Hence it follows by Lemma 18 that the differential operator (55) multiplied by

$$(t_1t_2)^n b^n \prod_{p|b} p^{s_1\tau_p(1)+s_2\tau_p(2)+\dots+s_n\tau_p(n)} \tag{58}$$

takes the ring $\mathbb{Z}[y_1, \dots, y_m]$ into itself. Since $s_1 + 2s_2 + \dots + ns_n = n \leqslant k$, it follows from Lemma 19 that the integer (58) divides

$$\psi_k = (t_1t_2)^k b^k \prod_{p|b} p^{\tau_p(k)}.$$

We now apply the identity of Lemma 17. For the sequence $\{\psi_k\}_{k\in\mathbb{N}}$ the operators (53) take the ring $\mathbb{Z}[y_1, \dots, y_m]$ into itself. The estimate (18) completes the proof of the theorem.

Remark. Corresponding to the system of differential equations (8), (12) of Fuchs type is the differential operator

$$D = \frac{d}{dz} + \frac{1}{z-\gamma_1}[A_1] + \dots + \frac{1}{z-\gamma_s}[A_s].$$

This enables one to use Theorem 7 for another proof of Theorem 5 in the case of commuting matrices $A_1, \dots, A_s$ such that their minimal polynomials have no multiple zeros. In that way one obtains, however, a slightly worse constant Ψ (by comparison with our formulation of Theorem 5).

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