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Lower bounds for polynomials in the values of certain entire functions

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Abstract. Lower estimates in terms of all coefficients are established for polynomials and linear forms in the values of E-functions. Consequences for generalized hypergeometric E-functions are indicated.

Bibliography: 9 titles.

Introduction

History of the problem. An important part of the theory of Diophantine approximations and transcendental numbers is the study of the behaviour of the value of

$$|h_1\xi_1 + \dots + h_m\xi_m| \qquad (h_j \in \mathbb{Z}, \quad j = 1, \dots, m) \tag{0.1}$$

for given real $\xi_1, \dots, \xi_m$ and the lower estimate of this quantity in terms of the integer coefficients $h_1, \dots, h_m$. As follows from Dirichlet's theorem (see, for example, [1], Chapter 1, § 2, Theorem 4) *for each real $H \geqslant 1$ there exist integers $h_1, \dots, h_m$ such that*

$$|h_1\xi_1 + \dots + h_m\xi_m| < H^{-m+1} \quad \text{and} \quad 0 < \max_{1\leqslant j\leqslant m}\{|h_j|\} \leqslant H.$$

Dirichlet's theorem answers thereby the following question: how small can be (0.1) for a given value of

$$\max_{1\leqslant j\leqslant m}\{|h_j|\}?$$

In doing so it is completely indifferent to the nature of $\xi_1, \dots, \xi_m$. By metric considerations ([2], Chapter I, Theorem 12), *given an arbitrary $\varepsilon > 0$ for almost all (in the sense of Lebesgue measure) points $\bar\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m$ there exists a constant $C = C(\bar\xi, \varepsilon) > 0$ such that for arbitrary integers $h_1, \dots, h_m$ that do not all vanish simultaneously*

$$|h_1\xi_1 + \dots + h_m\xi_m| > C(H_1 \cdots H_m)^{-1} H(\log H)^{-m+1-\varepsilon},$$

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where

$$H_j = \max\{1, |h_j|\}, \quad j = 1, \dots, m, \quad \text{and} \quad H = \max\Bigl\{e, \max_{1\leqslant j\leqslant m}\{H_j\}\Bigr\}.$$

However, no particular collection of numbers $\bar{\xi}$ such that this inequality holds is known so far.

Using the methods of the theory of transcendental numbers one can obtain for (0.1) lower estimates

$$|h_1\xi_1 + \dots + h_m\xi_m| > CH^{-m+1-\varepsilon}$$

for certain particular choices of $\bar{\xi}$. At the same time, in several papers related (for example) to the estimates of the deviations of uniformly distributed sequences the authors require lower estimates of the absolute values of linear forms (0.1), and these must be estimates in terms of all the coefficients. The following result of Baker [3] for the values of the exponential function was apparently the first of this kind:

$$|h_1e^{\alpha_1} + \dots + h_me^{\alpha_m}| > C(H_1 \cdots H_m)H^{1-\gamma/\sqrt{\log\log H}}.$$

To show this he needed to improve somewhat the method proposed by Siegel [4]. However, Baker's scheme has never been significantly generalized (an endeavour in that direction was presented in [5]).

There is a natural generalization of the problem of finding lower estimates for linear forms in real numbers. Namely, one can study the behaviour of the quantity

$$|P(\xi_1, \dots, \xi_m)|, \qquad P \in \mathbb{Z}[y_1, \dots, y_m], \tag{0.2}$$

in its dependence on the coefficients of the polynomial $P(y_1, \dots, y_m)$ (in particular, on the height $H = H(P)$ of this polynomial) and on its degree $d = \deg P$.

The above-mentioned Siegel's method enable one to carry this out in the case when $\xi_1, \dots, \xi_m$ are the values at an algebraic point $\alpha \in \mathbb{K} \setminus \{0\}$ of the analytic functions

$$f_j(z) = \sum_{\nu=0}^{\infty} f_{j,\nu}z^\nu, \qquad f_{j,\nu} \in \mathbb{K}, \quad j = 1, \dots, m, \quad \nu \in \mathbb{Z}^+ = \{0, 1, 2, \dots\}, \tag{0.3}$$

that, combined, are the components of a solution of a system of linear differential equations of the first order

$$\begin{gathered} \frac{d}{dz}y_l = \sum_{j=1}^{m} Q_{lj}y_j, \qquad l = 1, \dots, m, \\ Q_{lj} = Q_{lj}(z) \in \mathbb{C}(z), \qquad l, j = 1, \dots, m. \end{gathered} \tag{0.4}$$

In addition, the Taylor coefficients of the functions (0.3) must satisfy certain additional arithmetic conditions describing the class of E-functions and α may not be a singular point of (0.4). Siegel's method was considerably refined by Shidlovskiĭ who established, in particular, a test for the algebraic independence of the values of E-functions. (See his monograph [1] for a detailed history of this problem.) We note at the onset that, throughout, only the case of $\mathbb{K} = \mathbb{Q}$ will be considered because all the estimates in this case have their natural counterparts for an arbitrary finite extension of the rationals.

Definition [6]**.** We call a function

$$f(z)=\sum_{\nu=0}^{\infty}\frac{f_\nu}{\nu!}z^\nu,\qquad f_\nu\in\mathbb{Q},\quad \nu\in\mathbb{Z}^+,$$

an E*-function* if there exists a positive constant C such that $|f_\nu|<C^{\nu+1}$ for $\nu\in\mathbb{Z}^+$ and there exists a sequence of positive integers $\{\varphi_n\}_{n=1}^{\infty}$ such that $\varphi_n<C^n$ and $\varphi_n f_\nu\in\mathbb{Z}$ for $\nu=0,1,\dots,n$ and $n\in\mathbb{N}$.

This slightly differs from Siegel's classical definition. However, this definition covers all known E-functions in Siegel's sense that have rational Taylor coefficients and are solutions of linear differential equations. In particular, this holds for all entire hypergeometric functions with rational parameters (see [1], Chapter 5, § 1).

In 1984, Chudnovsky [7] put forward an ingenious construction. It enables one to obtain lower estimates

$$|h_1f_1(\alpha)+\dots+h_mf_m(\alpha)|>C(H_1\cdots H_m)^{-1}H^{1-\varepsilon}$$

for linear forms in the values of E-functions (0.3) satisfying their own linear homogeneous differential equations of arbitrary orders. He imposed a very stringent constraint on the set of the equations in question, which was similar to Siegel's condition. In addition, Chudnovsky was short of passing consistently from linear approximations of functional forms (which he called *graded Padé approximations* to numerical linear forms. By and large, Chudnovsky's method was a direct development of that of Siegel and Shidlovskiĭ. Further generalization of the scheme of graded Padé approximations [8] resulted in the following, exact in order, irrationality measure of the values of E-functions (0.3):

$$\left|f_l(\alpha)-\frac{p}{q}\right|>|q|^{-2-\gamma(\log\log|q|)^{-1/(m+1)}},\qquad l=1,\dots,m,$$

with an easily verified condition on the class of functions in question.

In the present paper, by introducing new ideas into the Siegel–Shidlovskiĭ method we implement the construction of graded Padé approximations in its full extent and deduce lower estimates of the quantities (0.1) and (0.2). Moreover, our condition on the class of functions under discussion is more simple than Chudnovsky's and has been verified in many cases. This enables us to deduce several consequences for the generalized hypergeometric functions.

Main results. We say that *the system* (0.4) *of linear homogeneous differential equations of the first order is in the class* $\mathbf{W}^0$ if the entries of some fundamental matrix $(\psi_{jl})_{j,l=1,\dots,m}$ of solutions of (0.4) are homogeneously algebraically independent over $\mathbb{C}(z)$. We note that the phrase 'some fundamental matrix' can be replaced in this case by 'an arbitrary fundamental matrix', for all such matrices differ by a constant matrix factor.

Theorem I. *Let* $f_1(z),\dots,f_m(z)$, $m\geqslant 2$, *be a collection of* E*-functions satisfying the system* (0.4) *of linear homogeneous differential equations that is in the class* $\mathbf{W}^0$, *let* $\alpha\in\mathbb{Q}\setminus\{0\}$ *be a non-singular point of this system, and let* $d\in\mathbb{N}$*. Then there* *exist positive constants* $\gamma = \gamma(f_1, \dots, f_m; \alpha, d)$ *and* $C = C(f_1, \dots, f_m; \alpha, d)$ *such that*

$$\bigl|P\bigl(f_1(\alpha), \dots, f_m(\alpha)\bigr)\bigr| > C \cdot |h_1 \cdots h_w|^{-1} H^{1-\gamma(\log\log H)^{-1/(m^2-m+2)}}, \qquad H = \max_{1 \leqslant i \leqslant w} \bigl\{|h_i|\bigr\} \geqslant 3$$

for each homogeneous polynomial $P \in \mathbb{Z}[y_1, \dots, y_m]$ *of degree* d*, where* $h_1, \dots, h_w$ *are all the non-zero coefficients of* $P(y_1, \dots, y_m)$*.*

By Theorem I we immediately obtain the following result on lower estimates for linear forms in the values of E-functions.

Corollary. *Let* $f_1(z), \dots, f_m(z)$*,* $m \geqslant 2$*, be a collection of* E*-functions satisfying the system* (0.4) *of linear homogeneous differential equations that is in the class* $\mathbf{W}^0$*, and let* $\alpha \in \mathbb{Q} \setminus \{0\}$ *be a non-singular point of this system. Then there exists positive constants* $C = C(f_1, \dots, f_m; \alpha)$ *and* $\gamma = \gamma(f_1, \dots, f_m; \alpha)$ *such that*

$$|h_1 f_1(\alpha) + \dots + h_m f_m(\alpha)| > C \cdot (H_1 \cdots H_m)^{-1} H^{1-\gamma(\log\log H)^{-1/(m^2-m+2)}},$$
$$h_i \in \mathbb{Z}, \quad \text{where} \quad H_i = \max\bigl\{1, |h_i|\bigr\}, \quad i = 1, \dots, m, \quad \text{and} \quad H = \max_{1 \leqslant i \leqslant m} \{H_i\} \geqslant 3.$$

In actual fact, Theorem I is a particular case of a certain more general result that we prove below. To formulate it we need another concept. In what follows we consider systems of linear homogeneous differential equations of the first order split into m subsystems

$$\begin{aligned} \frac{d}{dz} y_{il} &= \sum_{j=1}^{m_i} Q_{lj}^{(i)} y_{ij}, \qquad l = 1, \dots, m_i, \\ Q_{lj}^{(i)} &= Q_{lj}^{(i)}(z) \in \mathbb{Q}(z), \qquad l, j = 1, \dots, m_i, \end{aligned} \qquad i = 1, \dots, m, \quad m \geqslant 2. \tag{0.5}$$

If $\alpha \in \mathbb{C}$ is a regular point of (0.5), then it is also regular for the conjugate system of linear homogeneous differential equations

$$\frac{d}{dz} a_{ij} = -\sum_{l=1}^{m_i} Q_{lj}^{(i)} a_{il}, \qquad j = 1, \dots, m_i, \quad i = 1, \dots, m. \tag{0.6}$$

Hence there exists a collection of analytic functions

$$\varphi_{ij} = \varphi_{ij}(z), \qquad j = 1, \dots, m_i, \quad i = 1, \dots, m, \tag{0.7}$$

in a neighbourhood of $z = \alpha$ such that these functions solve (0.6) and

$$\varphi_{ij}(\alpha) = \begin{cases} 1 & \text{for } j = 1, \\ 0 & \text{for } j = 2, \dots, m_i, \end{cases} \qquad i = 1, \dots, m. \tag{0.8}$$

If the functions (0.7) are homogeneously algebraically independent over $\mathbb{C}(z)$, then we say that *the system* (0.5) *is in the class* $\mathbf{W}^0(\alpha)$.

Theorem II. *Let*

$$f_{il}(z), \qquad l=1,\dots,m_i, \quad i=1,\dots,m, \tag{0.9}$$

be a collection of E-functions satisfying the system (0.5) *of linear homogeneous differential equations in the class* $\mathbf{W}^0(\alpha)$ ($\alpha\in\mathbb{Q}\setminus\{0\}$). *For an arbitrary homogeneous polynomial $P=P(y_1,\dots,y_m)\in\mathbb{Z}[y_1,\dots,y_m]$ of degree $d\in\mathbb{N}$ let U be the set of multi-indices such that*

$$\begin{gathered} P(y_1,\dots,y_m)=\sum_{\bar u\in U} h_{\bar u} y_1^{u_1}\cdots y_m^{u_m}, \\ \text{where} \quad h_{\bar u}\neq 0 \quad \text{and} \quad |\bar u|=u_1+\dots+u_m=d \quad \text{for } \bar u\in U. \end{gathered} \tag{0.10}$$

Assume also that the functions

$$F_{\bar u}(z)=f_{11}^{u_1}(z)f_{21}^{u_2}(z)\cdots f_{m1}^{u_m}(z), \qquad \bar u\in U, \tag{0.11}$$

are linearly independent over $\mathbb{C}(z)$. Then there exist positive constants γ and C dependent only on the collection (0.9), *d, and the regular point α such that*

$$\bigl|P\bigl(f_{11}(\alpha),f_{21}(\alpha),\dots,f_{m1}(\alpha)\bigr)\bigr| > C\prod_{\bar u\in U}|h_{\bar u}|^{-1}H^{1-\gamma(\log\log H)^{-1/(m_1+\dots+m_m-m+2)}},$$
$$H=\max_{\bar u\in U}\bigl\{|h_{\bar u}|\bigr\}\geqslant 3.$$

Proof of Theorem I. If $f_1(z),\dots,f_m(z)$ is the collection from the hypotheses of Theorem I, then we consider the m 'rolled' copies of this collection

$$f_{il}(z)\equiv\begin{cases} f_{i-l+1}(z), & \text{for } i-l\geqslant 0,\\ f_{i-l+1+m}(z), & \text{for } i-l<0,\end{cases} \qquad i,l=1,\dots,m,$$

every of which satisfies a system of linear homogeneous differential equations produced from (0.4) by means of some rearrangement of the subscripts. We note from the outset that since $f_1(z),\dots,f_m(z)$ are elements of some fundamental system of solutions of the system (0.4), which is in the class $\mathbf{W}^0$, they are homogeneously algebraically independent over $C(z)$.

Let

$$\psi_{lj}(z), \qquad l,j=1,\dots,m, \tag{0.12}$$

be the entries of the fundamental matrix of solutions of (0.4) that is equal to identity for $z=\alpha$ (that is,

$$\psi_{lj}(\alpha)=\delta_{lj}, \qquad l,j=1,\dots,m). \tag{0.13}$$

Then the functions (0.12) are homogeneously algebraically independent over $\mathbb{C}(z)$ because the system (0.4) is in $\mathbf{W}^0$. Hence the entries of the inverse matrix

$$\bigl(\widetilde{\psi}_{lj}(z)\bigr)_{l,j=1,\dots,m}=\bigl(\psi_{lj}(z)\bigr)_{l,j=1,\dots,m}^{-1},$$

which are rational functions of (0.12), are also homogeneously algebraically independent over $\mathbb{C}(z)$; moreover, in view of (0.13), they are normalized at the point $z=\alpha$, that is,

$$\widetilde{\psi}_{lj}(\alpha)=\delta_{lj}, \qquad l,j=1,\dots,m,$$

It remains to observe that the functions

$$\varphi_{ij}(z)=\begin{cases}\widetilde{\psi}_{i,j+i-1} & \text{for } j+i-1\leqslant m,\\ \widetilde{\psi}_{i,j+i-1-m} & \text{for } j+i-1>m,\end{cases} \qquad i,j=1,\dots,m,$$

are homogeneously algebraically independent over $\mathbb{C}(z)$, are solutions of the collection of systems conjugate to the systems corresponding to the $f_{il}(z)$, and satisfy (0.8). Hence the full system for $f_{il}(z)$ is in the class $\mathbf{W}^0(\alpha)$. The functions (0.11) are linearly independent over $\mathbb{C}(z)$ since the $f_i(z)\equiv f_{i1}(z)$, $i=1,\dots,m$, are algebraically independent. To complete the proof of Theorem I we now apply Theorem II.

Applications. To illustrate the applications of Theorem II we shall use the classical Siegel's result [4] on the values of the function

$$K_\lambda(z)=\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu!(\lambda+1)_\nu}\Big(\frac{z}{2}\Big)^{2\nu} \quad (\text{here} \quad \lambda\in\mathbb{Q}\setminus\{-1,-2,\dots\} \quad \text{and}$$
$$(\lambda+1)_0=1, \qquad (\lambda+1)_\nu=(\lambda+1)\cdots(\lambda+\nu), \quad \nu=1,2,\dots),$$

which satisfies the linear homogeneous differential equation of the second order

$$y''+\frac{2\lambda+1}{z}y'+y=0.$$

Theorem III. *Let λ_i, $i=1,\dots,m$, be a collection of numbers such that*

$$\lambda_i\in\mathbb{Q}\setminus\{-1,-2,\dots\}, \qquad \lambda_i\neq\frac{2\mu-1}{2}, \quad \mu\in\mathbb{Z}, \qquad i=1,\dots,m,$$
$$\lambda_{i_1}\pm\lambda_{i_2}\notin\mathbb{Z}, \qquad i_1,i_2=1,\dots,m, \quad i_1\neq i_2,$$

and let

$$\xi_1,\dots,\xi_n\in\mathbb{Q}\setminus\{0\}, \qquad \xi_{j_1}^2\neq\xi_{j_2}^2, \qquad j_1,j_2=1,\dots,n, \quad j_1\neq j_2.$$

Then there exist positive constants C and γ dependent only on the parameters $\lambda_1,\dots,\lambda_m$ and $\xi_1,\dots,\xi_n$ and a positive integer d such that if P is an arbitrary (not necessarily homogeneous) polynomial of degree d in y_{ij}, $i=1,\dots,m$, $j=1,\dots,n$, with integer coefficients, then

$$\Big|P\big|_{y_{ij}=K_{\lambda_i}(\xi_j)}\Big|>C|\Pi|^{-1}H^{1-\gamma(\log\log H)^{-1/(mn+2)}},$$

where Π *is the product of all the non-trivial coefficients of* P *and* $H \geqslant 3$ *is the height of the polynomial.*

Proof. The functions $y_1 = K_\lambda(\xi z)$ and $y_2 = K'_\lambda(\xi z)$ make up a solution of the system

$$\frac{d}{dz}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\xi^2 & -\frac{2\lambda+1}{z} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

of linear homogeneous differential equations. The corresponding conjugate system is as follows:

$$\frac{d}{dz}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 & \xi^2 \\ -1 & \frac{2\lambda+1}{z} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{0.14}$$

If $a_1(z), a_2(z)$ is a solution of (0.14), then the function $\varphi(z) = a_1(z/\xi^2)$ and its derivative $\psi(z) = \varphi'(z) = a_2(z/\xi^2)$ form a solution of the system

$$\frac{d}{dz}\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{\xi^2} & \frac{2\lambda+1}{z} \end{pmatrix}\begin{pmatrix} \varphi \\ \psi \end{pmatrix}.$$

Let $\varphi_{\lambda,\xi}(z)$ be a function $\varphi(z)$ of this kind satisfying the conditions $\varphi(\xi^2) = a_1(1) = 1$ and $\varphi'(\xi^2) = a_2(1) = 0$.

We now use Lemma 1 in [1], Chapter 9, § 1, which says that if the parameters $\lambda_1, \dots, \lambda_m$ and $\xi_1, \dots, \xi_n$ are as in the hypotheses of the theorem, then the functions

$$\varphi_{\lambda_i,\xi_j}(z), \quad \varphi'_{\lambda_i,\xi_j}(z), \qquad i = 1, \dots, m, \quad j = 1, \dots, n,$$

are algebraically independent over $\mathbb{C}(z)$, which means that the system of differential equations with solutions

$$f_0(z) = 1, \ f_{ij,1}(z) = K_{\lambda_i}(\xi_j z), \ f_{ij,2}(z) = K'_{\lambda_i}(\xi_j z), \quad i = 1, \dots, m, \quad j = 1, \dots, n,$$

is in the class $\mathbf{W}^0(\alpha)$. To complete the proof of Theorem III we now use Theorem II.

Remark. The condition that (0.6) be in the class $\mathbf{W}^0(\alpha)$ for a given regular point $z = \alpha$ is weaker than Siegel's condition of normality for this system.

§ 1. Auxiliary results

Ranks of special numerical linear forms. Let $\mathbb{M}$ be the module of linear forms in $y_1, \dots, y_m$ over $\mathbb{C}[z]$, so that the elements $R \in \mathbb{M}$ are of the following form:

$$R = \sum_{k=1}^{m} P_k(z) y_k, \qquad P_k(z) \in \mathbb{C}[z], \quad k = 1, \dots, m.$$

We now consider the system

$$y'_l = \sum_{j=1}^{m} Q_{lj} y_j, \qquad Q_{lj} = Q_{lj}(z) \in \mathbb{C}(z), \quad l, j = 1, \dots, m, \tag{1.1}$$

of linear homogeneous differential equations of the first order. We choose a polynomial $T = T(z)$ such that $TQ_{lj} \in \mathbb{C}[z]$ for $l, j = 1, \dots, m$.

We consider the differential operator

$$D = \frac{\partial}{\partial z} + \sum_{l=1}^{m} \biggl(\sum_{j=1}^{m} Q_{lj} y_j \biggr) \frac{\partial}{\partial y_l}, \tag{1.2}$$

on $\mathbb{M}$, which is connected with (1.1). If $R \in \mathbb{M}$, then, clearly, also $TDR \in \mathbb{M}$. Hence TD acts from $\mathbb{M}$ into $\mathbb{M}$. Moreover (see [1], Chapter 3, § 4), if $y_1, \dots, y_m$ is a solution of (1.1), then

$$DR = \frac{d}{dz} R = R'.$$

Now let

$$R^{[0]} = \sum_{k=1}^{m} P_k^{[0]}(z) y_k, \qquad P_k^{[0]}(z) \in \mathbb{C}[z], \quad \text{where} \quad k = 1, \dots, m, \tag{1.3}$$

be an arbitrary linear form in $\mathbb{M}$ and let

$$R^{[n+1]} = TDR^{[n]}, \qquad n \geqslant 0. \tag{1.4}$$

Then, according to the above, $R^{[n]} \in \mathbb{M}$ for $n \geqslant 1$, that is,

$$R^{[n]} = \sum_{k=1}^{m} P_k^{[n]}(z) y_k, \quad \text{where} \quad P_k^{[n]}(z) \in \mathbb{C}[z], \qquad n \geqslant 0, \quad k = 1, \dots, m.$$

Using the definition (1.2) of D and equality (1.4) we obtain the following recursion relations:

$$P_k^{[n+1]}(z) = T(z) \biggl(\frac{d}{dz} P_k^{[n]}(z) + \sum_{l=1}^{m} P_l^{[n]}(z) Q_{lk}(z) \biggr), \qquad n \geqslant 0, \quad k = 1, \dots, m. \tag{1.5}$$

We now set $\Omega = \{1, \dots, m\}$. In this subsection we prove the following result.

Proposition 1.1. *For a linear form* (1.3) *in* $\mathbb{M}$ *assume that the square matrix*

$$\bigl(P_k^{[n]}(z) \bigr)_{n=0,1,\dots,m-1; k \in \Omega} \tag{1.6}$$

with entries defined by (1.5) *has rank precisely* $\widetilde{m}$ *over* $\mathbb{C}(z)$. *For arbitrary* $\widetilde{\Omega} \subset \Omega$, $\operatorname{Card} \widetilde{\Omega} = \widetilde{m}$, *let*

$$\Delta(\widetilde{\Omega}; z) = \det \bigl(P_k^{[n]}(z) \bigr)_{n=0,1,\dots,\widetilde{m}-1; k \in \widetilde{\Omega}}.$$

Let $\alpha \in \mathbb{C}$ *be a regular point of* (1.1) (*that is,* $T(\alpha) \neq 0$) *and assume that*

$$\max_{\widetilde{\Omega}} \Bigl\{ \operatorname*{ord}_{z=\alpha} \Delta(\widetilde{\Omega}; z) \Bigr\} = q,$$

where the maximum is considered over all the subsets $\widetilde{\Omega}\subset\Omega$ *such that* $\operatorname{Card}\widetilde{\Omega}=\widetilde{m}$ *and* $\Delta(\widetilde{\Omega};z)\not\equiv 0$. *Then the rank over* $\mathbb{C}$ *of the numerical matrix*

$$\bigl(P_k^{[n]}(\alpha)\bigr)_{n=0,1,\dots,\widetilde{m}+q-1;k\in\Omega}$$

is precisely equal to $\widetilde{m}$.

Remark. In what follows we actually require the scheme of the proof of this result rather than the result itself. Still, we believe that Proposition 1.1 is a fairly interesting and useful generalization of a lemma of Siegel (see [1], Chapter 3, § 7, Lemma 10).

Proof. By Lemma 7 in [1], Chapter 3, § 4 we can choose the fundamental system of solutions

$$\bigl(y_{k\eta}(z)\bigr)_{k\in\Omega;\eta=1,\dots,m} \tag{1.7}$$

of (1.1) such that the forms $R^{[0]},R^{[1]},R^{[2]},\dots$ vanish for $y_1,\dots,y_m$ set to be equal to any of the $m-\widetilde{m}$ solutions $y_{1\eta},\dots,y_{m\eta}$, $\eta=\widetilde{m}+1,\dots,m$. We note from the outset that all the entries of (1.7) are analytic functions at $z=\alpha$ since this is a regular point of (1.1).

For the result of the substitution of the $y_{1\eta},\dots,y_{m\eta}$ ($\eta=1,\dots,m$) in $R^{[n]}\in\mathbb{M}$ for $y_1,\dots,y_m$ we use the notation

$$R_\eta^{[n]}=R_\eta^{[n]}(z)=\sum_{k\in\Omega}P_k^{[n]}(z)y_{k\eta}(z),\qquad n\geqslant 0,\quad \eta=1,\dots,\omega. \tag{1.8}$$

Then by our choice of (1.7),

$$R_\eta^{[n]}(z)\equiv 0,\qquad n\geqslant 0,\quad \eta=\widetilde{m}+1,\dots,m. \tag{1.9}$$

We now consider the following analytic functions in a neighbourhood of $z=\alpha$:

$$\Lambda(z)=\det\bigl(y_{k\eta}(z)\bigr)_{k\in\Omega;\eta=1,\dots,m}\quad\text{and}$$
$$\lambda(\widetilde{\Omega};z)=\det\bigl(y_{k\eta}(z)\bigr)_{k\in\Omega\setminus\widetilde{\Omega};\eta=\widetilde{m}+1,\dots,m},\quad\text{where}\quad \widetilde{\Omega}\subset\Omega,\quad \operatorname{Card}\widetilde{\Omega}=\widetilde{m}$$

(if $\widetilde{m}=m$, then we set $\lambda(\Omega;z)\equiv 1$).

Lemma 1.2. *Let* $\mathcal{N}\subset\mathbb{Z}^+=\{0,1,2,\dots\}$ *be an arbitrary set such that* $\operatorname{Card}\mathcal{N}=\widetilde{m}$ *and let* $\widetilde{\Omega}\subset\Omega$, $\operatorname{Card}\widetilde{\Omega}=\widetilde{m}$. *Then*

$$\det\bigl(P_k^{[n]}(z)\bigr)_{n\in\mathcal{N};k\in\widetilde{\Omega}}\cdot\Lambda(z)=\det\bigl(R_\eta^{[n]}(z)\bigr)_{n\in\mathcal{N};\eta=1,\dots,\widetilde{m}}\cdot\lambda(\widetilde{\Omega};z). \tag{1.10}$$

Proof. Multiplying matrices and using the notation (1.8) we obtain

$$\begin{pmatrix}P_k^{[n]}(z)\\ \delta_{ik}\end{pmatrix}_{n\in\mathcal{N},i\in\Omega\setminus\widetilde{\Omega};k\in\Omega}\bigl(y_{k\eta}(z)\bigr)_{k\in\Omega;\eta=1,\dots,m}$$
$$=\begin{pmatrix}R_\eta^{[n]}(z)\\ y_{i\eta}(z)\end{pmatrix}_{n\in\mathcal{N},i\in\Omega\setminus\widetilde{\Omega};\eta=1,\dots,m}, \tag{1.11}$$

where δ_{ik} is the Kronecker delta. By (1.9),

$$\det\begin{pmatrix} R_\eta^{[n]}(z) \\ y_{i\eta}(z) \end{pmatrix}_{n\in\mathcal{N}, i\in\Omega\setminus\widetilde{\Omega};\eta=1,\dots,m} = \det\bigl(R_\eta^{[n]}(z)\bigr)_{n\in\mathcal{N};\eta=1,\dots,\widetilde{m}} \cdot \det\bigl(y_{i\eta}(z)\bigr)_{i\in\Omega\setminus\widetilde{\Omega};\eta=\widetilde{m}+1,\dots,m};$$

moreover,

$$\det\begin{pmatrix} P_k^{[n]}(z) \\ \delta_{ik} \end{pmatrix}_{n\in\mathcal{N}, i\in\Omega\setminus\widetilde{\Omega};k\in\Omega} = \det\bigl(P_k^{[n]}(z)\bigr)_{n\in\mathcal{N};k\in\widetilde{\Omega}}.$$

Hence we obtain (1.10) by passing from the matrices in (1.11) to their determinants.

Lemma 1.3. *Let $\widetilde{\Omega}\subset\Omega$, $\operatorname{Card}\widetilde{\Omega}=\widetilde{m}$, be a subset such that $\lambda(\widetilde{\Omega};\alpha)\ne 0$. Then $\Delta(\widetilde{\Omega};z)\not\equiv 0$. If in addition*

$$\operatorname*{ord}_{z=\alpha}\Delta(\widetilde{\Omega};z)=p,$$

then the rank of the numerical matrix

$$\bigl(P_k^{[n]}(\alpha)\bigr)_{n=0,1,\dots,\widetilde{m}+p-1;k\in\widetilde{\Omega}} \tag{1.12}$$

is equal to $\widetilde{m}$.

Proof. For $\mathcal{N}=\{0,1,\dots\widetilde{m}-1\}$,

$$\Delta(\widetilde{\Omega};z)\cdot\Lambda(z)=\det\bigl(R_\eta^{[n]}(z)\bigr)_{n=0,1,\dots,\widetilde{m}-1;\eta=1,\dots,\widetilde{m}}\cdot\lambda(\widetilde{\Omega};z) \tag{1.13}$$

by the identity in Lemma 1.2. Assuming that $\lambda(\widetilde{\Omega};z)\not\equiv 0$, while $\Delta(\widetilde{\Omega};z)\equiv 0$, we obtain that $\det\bigl(R_\eta^{[n]}(z)\bigr)_{n=0,1,\dots,\widetilde{m}-1;\eta=1,\dots,\widetilde{m}}\equiv 0$. Hence we see from (1.13) that $\Delta(\widetilde{\Omega};z)\equiv 0$ for each $\widetilde{\Omega}\subset\Omega$ such that $\operatorname{Card}\widetilde{\Omega}=\widetilde{m}$. In other words, the rank of the matrix

$$\bigl(P_k^{[n]}(z)\bigr)_{n=0,1,\dots,\widetilde{m}-1;k\in\Omega} \tag{1.14}$$

is smaller than $\widetilde{m}$. On the other hand, the rank of (1.6) is $\widetilde{m}$, therefore by Lemma 6 in [1], Chapter 3, § 4, the rank of (1.14) is also $\widetilde{m}$. This contradiction shows that if $\lambda(\widetilde{\Omega};z)\not\equiv 0$, then $\Delta(\widetilde{\Omega};z)\not\equiv 0$.

We now proceed to the proof of the second part of the lemma. For a set $\widetilde{\Omega}$ let $\Delta(z)=\Delta(\widetilde{\Omega};z)$. We now rewrite (1.13) as follows:

$$\Delta(z)\cdot\chi(z)=\det\bigl(R_\eta^{[n]}(z)\bigr)_{n=0,1,\dots,\widetilde{m}-1;\eta=1,\dots,\widetilde{m}}, \tag{1.15}$$

where $\chi(z)=\Lambda(z)/\lambda(\widetilde{\Omega};z)$ is a function analytic at $z=\alpha$ because $\lambda(\widetilde{\Omega};\alpha)\ne 0$. We also rewrite (1.4) for the functions (1.8) as follows:

$$R_\eta^{[n+1]}(z)=T(z)\frac{d}{dz}R_\eta^{[n]}(z), \qquad n\geqslant 0, \quad \eta=1,\dots,\widetilde{m}. \tag{1.16}$$

Using (1.15), (1.16), the differentiation rules for a determinant regarded as a function of rows, and the identity in Lemma 1.2 we obtain

$$\begin{aligned}\Bigl(T(z)\frac{d}{dz}\Bigr)^p\bigl(\Delta(z)\chi(z)\bigr) &= \Bigl(T(z)\frac{d}{dz}\Bigr)^p \det\bigl(R_\eta^{[n]}(z)\bigr)_{n=0,1,\dots,\widetilde{m}-1;\eta=1,\dots,\widetilde{m}} \\ &= \sum_{\substack{\nu_0+\dots+\nu_{\widetilde{m}-1}=p\\ \nu_0,\dots,\nu_{\widetilde{m}-1}\geqslant 0}} \frac{p!}{\nu_0!\cdots\nu_{\widetilde{m}-1}!}\det\bigl(R_\eta^{[n+\nu_n]}(z)\bigr)_{n=0,1,\dots,\widetilde{m}-1;\eta=1,\dots,\widetilde{m}} \\ &= \sum_{\substack{\nu_0+\dots+\nu_{\widetilde{m}-1}=p\\ \nu_0,\dots,\nu_{\widetilde{m}-1}\geqslant 0}} \frac{p!}{\nu_0!\cdots\nu_{\widetilde{m}-1}!}\det\bigl(P_k^{[n+\nu_n]}(z)\bigr)_{n=0,1,\dots,\widetilde{m}-1;k\in\widetilde{\Omega}}\cdot\chi(z). \qquad (1.17)\end{aligned}$$

We note that for $\widetilde{m} = m$ the last identity was proved by Titenko (1987) in his diploma work. The function $\chi(z)$ is analytic and non-vanishing at $z = \alpha$; by assumption $T(\alpha) \neq 0$. Hence if p is the precise order of the zero of $\Delta(z)$ at $z = \alpha$, then setting $z = \alpha$ in (1.17) and dividing both sides by $\chi(\alpha)$ we obtain

$$0 \neq T^p(\alpha)\Delta^{(p)}(\alpha) = \sum_{\substack{\nu_0+\dots+\nu_{\widetilde{m}-1}=p\\ \nu_0,\dots,\nu_{\widetilde{m}-1}\geqslant 0}} \frac{p!}{\nu_0!\cdots\nu_{\widetilde{m}-1}!}\det\bigl(P_k^{[n+\nu_n]}(\alpha)\bigr)_{n=0,1,\dots,\widetilde{m}-1;k\in\widetilde{\Omega}}.$$

Hence there exists a collection $\nu_0,\dots,\nu_{\widetilde{m}-1}$ of non-negative integers such that $\nu_0 + \dots + \nu_{\widetilde{m}-1} = p$ and

$$\det\bigl(P_k^{[n+\nu_n]}(\alpha)\bigr)_{n=0,1,\dots,\widetilde{m}-1;k\in\widetilde{\Omega}} \neq 0,$$

which means, for its part, that the rank of (1.12) is precisely $\widetilde{m}$.

We now return to the proof of Proposition 1.1.

Assume that $\lambda(\widetilde{\Omega};\alpha) = 0$ for each $\widetilde{\Omega} \subset \Omega$ such that $\operatorname{Card}\widetilde{\Omega} = \widetilde{m}$. Then, by the definition of the $\lambda(\widetilde{\Omega};z)$ the numerical matrix $\bigl(y_{k\eta}(\alpha)\bigr)_{k\in\widetilde{\Omega};\eta=\widetilde{m}+1,\dots,m}$ has linearly dependent columns. However, this contradicts the assumptions that (1.7) is the fundamental matrix of solutions of (1.1) and $z = \alpha$ is a regular point of this system. Hence $\lambda(\widetilde{\Omega};\alpha) \neq 0$ for at least one such subset $\widetilde{\Omega}$. It remains to use Lemma 1.3 and the inequality $p \leqslant q$.

Galochkin's lemma. The result presented below is one of the key points of the method we present. We became aware of it courtesy A. I. Galochkin.

Lemma 1.4 (Galochkin). *Let*

$$\Delta(z) = \sum_{\nu=p}^{s} \frac{\Delta_\nu}{\nu!} z^\nu, \quad \textit{where} \quad \Delta_\nu \in \mathbb{Z}, \quad |\Delta_\nu| \leqslant \delta, \quad \nu = p, p+1, \dots, s,$$

$$s = \deg\Delta(z), \quad p = \operatorname*{ord}_{z=0}\Delta(z) \geqslant 2, \quad q = \operatorname*{ord}_{z=\alpha}\Delta(z), \quad \alpha\in\mathbb{C}, \quad |\alpha| \leqslant \frac{p-\sqrt{p}}{1+\sqrt{p}}.$$

Then

$$q \leqslant \frac{2\log(s\delta)}{\log p}. \tag{1.18}$$

Proof. For $|\xi| = p$,

$$|\Delta(\xi)| \leqslant (s-p+1)\frac{\delta}{p!}p^p$$

(here we use the inequality

$$\frac{p^\nu}{\nu!} \leqslant \frac{p^p}{p!} \quad \text{for } \nu \geqslant p).$$

Since

$$\Delta(z) = \frac{1}{2\pi i}\oint_{|\xi|=p} \frac{z^p(z-\alpha)^q}{\xi^p(\xi-\alpha)^q}\,\frac{\Delta(\xi)}{\xi - z}\,d\xi$$

and

$$\frac{s-p+1}{p-1} \leqslant \frac{s}{p},$$

it follows that for $|z| = 1$,

$$\begin{aligned}|\Delta(z)| &\leqslant \frac{1}{2\pi}\oint_{|\xi|=p} \frac{(1+|\alpha|)^q}{p^p(p-|\alpha|)^q}\,\frac{s-p+1}{p-1}\,\frac{\delta}{p!}p^p\,d\xi \leqslant s\delta\biggl(\frac{1+|\alpha|}{p-|\alpha|}\biggr)^q\frac{1}{p!}\\ &\leqslant s\delta\biggl(\frac{1}{\sqrt{p}}\biggr)^q\frac{1}{p!}.\end{aligned}$$

Further,

$$\Delta_p = \frac{p!}{2\pi i}\oint_{|z|=1}\frac{\Delta(z)}{z^{p+1}}\,dz,$$

therefore

$$|\Delta_p| \leqslant p!\cdot\max_{|z|=1}|\Delta(z)| \leqslant s\delta p^{-q/2}.$$

On the other hand, Δ_p is a non-zero integer, so that $|\Delta_p| \geqslant 1$ and therefore

$$p^{q/2} \leqslant s\delta.$$

Taking the logarithm we obtain (1.18), which completes the proof.

We call the following representation of a function $g(z)$ by a power series:

$$g(z) = \sum_{\nu=0}^{\infty}\frac{g_\nu}{\nu!}z^\nu, \qquad g_\nu \in \mathbb{C}, \tag{1.19}$$

the *normal expansion* of g and we call the numbers g_ν, $\nu \in \mathbb{Z}^+$, the *coefficients in the normal expansion* of g. If $g(z)$ is a polynomial, that is, the power series (1.19) contains only finitely many terms, then we set

$$\|g(z)\| = \max_\nu\bigl\{|g_\nu|\bigr\}.$$

Lemma 1.5 [8]**.** *Let $g(z)$ be a polynomial. Then*

(a) $\|g'(z)\| \leqslant \|g(z)\|$;
(b) $\|g_1(z)+g_2(z)\| \leqslant \|g_1(z)\| + \|g_2(z)\|$;
(c) $\|g_1(z)g_2(z)\| \leqslant \binom{\deg g_1(z)+\deg g_2(z)}{\deg g_1(z)} \|g_1(z)\| \cdot \|g_2(z)\|$.

We now present a consequence of Lemma 1.4, which is required in what follows.

Lemma 1.6. *Let*

$$\Delta(z) = \det\bigl(P_{nk}(z)\bigr)_{n,k=1,\dots,\widetilde{m}},$$

where the $P_{nk}(z) \in \mathbb{C}[z]$ are polynomials with integer coefficients in the normal expansion and

$$\deg P_{nk}(z) \leqslant d, \quad \|P_{nk}(z)\| \leqslant H, \qquad n,k=1,\dots,\widetilde{m}.$$

Assume that

$$\operatorname*{ord}_{z=0} \Delta(z) \geqslant p,$$

where p is sufficiently large. Let $z=\alpha$ be a fixed point. Then

$$\operatorname*{ord}_{z=\alpha} \Delta(z) < \frac{2\widetilde{m}}{\log p}\bigl(\log H + 2d(1+\log \widetilde{m})\bigr).$$

Proof. Expanding $\Delta(z)$ by the formula for a determinant, by Lemma 1.5 we obtain

$$\begin{aligned}\|\Delta(z)\| &\leqslant \sum_{\sigma\in \mathbf{S}_{\widetilde{m}}} \|P_{1\sigma(1)}(z)P_{2\sigma(2)}(z)\cdots P_{\widetilde{m}\sigma(\widetilde{m})}(z)\| \leqslant \widetilde{m}! \cdot \frac{(\widetilde{m}d)!}{(d!)^{\widetilde{m}}} \cdot H^{\widetilde{m}} \\ &\leqslant \widetilde{m}^{\widetilde{m}} \cdot \frac{(\widetilde{m}d)^{\widetilde{m}d}}{(d/e)^{\widetilde{m}d}} \cdot H^{\widetilde{m}} = \widetilde{m}^{\widetilde{m}(d+1)} e^{\widetilde{m}d} H^{\widetilde{m}}.\end{aligned}$$

It remains to use Lemma 1.4 and the fact that $\deg \Delta(z) \leqslant \widetilde{m}d < e^{\widetilde{m}d}$ to obtain

$$\begin{aligned}\operatorname*{ord}_{z=\alpha} \Delta(z) &\leqslant \frac{2\log(\deg \Delta(z)\cdot \|\Delta(z)\|)}{\log p} < \frac{2\log(\widetilde{m}^{\widetilde{m}(d+1)} e^{2\widetilde{m}d} H^{\widetilde{m}})}{\log p} \\ &\leqslant \frac{2\widetilde{m}}{\log p}\bigl(\log H + 2d(1+\log \widetilde{m})\bigr),\end{aligned}$$

as required.

§ 2. Graded Padé approximations

To prove Theorem II we now use the above-mentioned construction in [7].

Let $T(z)$ be the least common denominator of the rational coefficients in (0.5). Then

$$T(z) \in \mathbb{Z}[z] \quad \text{and} \quad T(z)Q_{lj}^{(i)}(z) \in \mathbb{Z}[z], \qquad l,j=1,\dots,m_i, \quad i=1,\dots,m. \tag{2.1}$$

The systems of approximating functional forms that we construct below depend on positive integer parameters $M_{\bar u}$, $\bar u\in U$. We set

$$
\begin{gathered}
M=\max_{\bar u\in U}\{M_{\bar u}\},\qquad N=\bigl[(\log M)^{1/(m_1+m_2+\dots+m_m-m+2)}\bigr]>d,\\
\varepsilon=\sum_{\bar u\in U}\sum_{i=1}^{m}\sum_{v=1}^{u_i}\frac{m_i-1}{N+m_i-v}\asymp\frac{1}{N},\qquad \min_{\bar u\in U}\{M_{\bar u}\}\geqslant 3\varepsilon M;
\end{gathered}
\tag{2.2}
$$

In addition we assume that M is sufficiently large. (Here and in what follows we denote by square brackets the integer part of a number.) We shall use letters C with subscripts and letters M with primes to denote positive constants that depend only on the functions (0.9), the systems (0.5), and the numbers α and d. We also use the notation $\bar a=(\bar a_1,\dots,\bar a_m)$, where $\bar a_i=(a_{i1},\dots,a_{im_i})$, $i=1,\dots,m$, and, in a similar way, we denote by $\bar\kappa=(\bar\kappa_1,\dots,\bar\kappa_m)$ multi-indices with $\bar\kappa_i=(\kappa_{i1},\dots,\kappa_{im_i})$, $i=1,\dots,m$. All the components of a multi-index must be non-negative, and if while considering a sum we come across a term with $\kappa_{ij}<0$ for some component of the multi-index, then this means that this term must be skipped (or vanishes). For reasons of space we shall also write

$$
\bar a^{\bar\kappa}=\prod_{\substack{i=1,\dots,m\\ j=1,\dots,m_i}}a_{ij}^{\kappa_{ij}}\quad\text{and}\quad|\bar\kappa_i|=\sum_{j=1}^{m_i}\kappa_{ij},\quad i=1,\dots,m.
$$

Let

$$
\begin{gathered}
\Omega_{\bar u}=\Omega_{\bar u}(N)=\{\bar\kappa:|\bar\kappa_i|=N-u_i,\ i=1,\dots,m\},\qquad \bar u\in U,\\
\Omega=\Omega(N)=\bigcup_{\bar u\in U}\Omega_{\bar u},\qquad \Theta=\Theta(N)=\{\bar s:|\bar s_i|=N,\ i=1,\dots,m\}.
\end{gathered}
$$

We shall use the appropriate small letters to denote the number of elements in these sets (cf. [1], Chapter 2, § 7, Lemma 7), that is,

$$
\begin{gathered}
\omega_{\bar u}=\operatorname{Card}\Omega_{\bar u}=\prod_{i=1}^{m}\binom{N+m_i-1-u_i}{m_i-1},\qquad \bar u\in U,\\
\omega=\operatorname{Card}\Omega=\sum_{\bar u\in U}\omega_{\bar u},\qquad \theta=\operatorname{Card}\Theta=\prod_{i=1}^{m}\binom{N+m_i-1}{m_i-1}.
\end{gathered}
$$

We shall be looking for linear forms of the following type:

$$
R(z;\bar a)=\sum_{\bar u\in U}P_{\bar u}(z;\bar a)\prod_{i=1}^{m}\bigl(a_{i1}f_{i1}(z)+\dots+a_{im_i}f_{im_i}(z)\bigr)^{N-u_i},
\tag{2.3}
$$

where $P_{\bar u}(z;\bar a)$ are polynomials of the following form, homogeneous in each component $\bar a_i$ of $\bar a$, $i=1,\dots,m$:

$$
P_{\bar u}(z;\bar a)=\sum_{\bar\kappa\in\Omega_{\bar u}}\bar a^{\bar\kappa}P_{\bar\kappa}(z),\qquad \bar u\in U.
\tag{2.4}
$$

We can represent the functional linear form (2.3) as follows:

$$R(z;\bar a)=\sum_{\bar s\in\Theta}\bar a^{\bar s}R_{\bar s}(z),\tag{2.5}$$

where

$$R_{\bar s}(z)=\sum_{\bar u\in U}\sum_{\substack{\bar r=(\bar r_1,\dots,\bar r_m)\\ |\bar r_i|=u_i,\,i=1,\dots,m}}\prod_{i=1}^{m}\frac{u_i!}{r_{i1}!\cdots r_{im_i}!}P_{\bar s-\bar r}(z)\bar f^{\,\bar r}(z),\qquad \bar s\in\Theta.\tag{2.6}$$

Lemma 2.1. *For positive integers $M_{\bar u}$, $\bar u\in U$, let the quantities M, N, and ε be defined in accordance with* (2.2). *Then there exist polynomials $P_{\bar\kappa}(z)\in\mathbb{Q}[z]$, $\bar\kappa\in\Omega$, such that*

(1) *these polynomials do not all vanish identically*;
(2) $\deg P_{\bar\kappa}<M$ *for all* $\bar\kappa\in\Omega$;
(3) $\operatorname{ord}_{z=0}P_{\bar\kappa}\geqslant M-M_{\bar u}$ *for all* $\bar\kappa\in\Omega_{\bar u}$ *and* $\bar u\in U$;
(4) *the coefficients in the normal expansions of these polynomials are integers with absolute values at most* $C_0^{M/\varepsilon}$;
(5) *the order of the zero at* $z=0$ *of each of the forms* (2.6) *is at least*

$$K=\Big[\sum_{\bar u\in U}\frac{\omega_{\bar u}}{\theta}M_{\bar u}-\varepsilon M\Big].$$

Proof. This can be proved using the same pattern as in the proof of Lemma 1.1 in [8].

Remark 1. We have

$$\begin{aligned}\frac{\omega_{\bar u}}{\theta}&=\prod_{i=1}^{m}\frac{\binom{N+m_i-1-u_i}{m_i-1}}{\binom{N+m_i-1}{m_i-1}}=\prod_{i=1}^{m}\frac{(N+m_i-1-u_i)!}{(N+m_i-1)!}\cdot\frac{N!}{(N-u_i)!}\\&=\prod_{i=1}^{m}\frac{(N-u_i+1)\cdots(N-1)N}{(N+m_i-u_i)\cdots(N+m_i-2)(N+m_i-1)}\\&=\prod_{i=1}^{m}\prod_{v=1}^{u_i}\Big(1-\frac{m_i-1}{N+m_i-v}\Big)>1-\sum_{i=1}^{m}\sum_{v=1}^{u_i}\frac{m_i-1}{N+m_i-v},\qquad \bar u\in U,\end{aligned}$$

therefore

$$\begin{aligned}K&=\Big[\sum_{\bar u\in U}\frac{\omega_{\bar u}}{\theta}M_{\bar u}-\varepsilon M\Big]=\Big[\sum_{\bar u\in U}M_{\bar u}-\sum_{\bar u\in U}\Big(1-\frac{\omega_{\bar u}}{\theta}\Big)M_{\bar u}-\varepsilon M\Big]\\&>\Big[\sum_{\bar u\in U}M_{\bar u}-\sum_{\bar u\in U}\sum_{i=1}^{m}\sum_{v=1}^{u_i}\frac{m_i-1}{N+m_i-v}M-\varepsilon M\Big]=\Big[\sum_{\bar u\in U}M_{\bar u}-2\varepsilon M\Big].\end{aligned}$$

The last inequality enables one to get a clearer idea of the value of K.

Remark 2. One must supplement Lemma 2.1 with the verification of the inequality $R(z;\bar a)\not\equiv 0$ for the form $R(z;\bar a)$ with coefficients constructed in this lemma. Considering all the multi-indices $\bar s\in\Theta$ such that the collection

$$P_{\bar s-\bar r}(z),\qquad |\bar r_i|=u_i,\quad i=1,\dots,m$$

contains at least one non-trivial polynomial we choose $\bar s'$ such that the sum $s'_{11}+s'_{21}+\dots+s'_{m1}$ is the largest possible. Then

$$R_{\bar s'}(z)=\sum_{\bar u\in U}P_{\bar s'-u_1\bar e_{11}-u_2\bar e_{21}-\dots-u_m\bar e_{m1}}(z)f_{11}^{u_1}(z)f_{21}^{u_2}(z)\cdots f_{m1}^{u_m}(z)\not\equiv 0$$

because the functions (0.11) are linearly independent over $\mathbb{C}(z)$ and in view of our choice of $\bar s'\in\Theta$. (Here we denote by $\bar e_{ij}$ the multi-index such that its component with subscript ij is equal to 1 and all the other are equal to 0.) Hence $R(z;\bar a)\not\equiv 0$.

Once we have constructed the form $R(z;\bar a)$ by Lemma 2.1, we shall produce more forms of this kind by means of the linear differential operator

$$D=\frac{\partial}{\partial z}-\sum_{i=1}^{m}\biggl(\sum_{j=1}^{m_i}\biggl(\sum_{l=1}^{m_i}Q_{lj}^{(i)}(z)a_{il}\biggr)\frac{\partial}{\partial a_{ij}}\biggr),$$

which is related to the collection of systems of linear homogeneous differential equations (0.6). We have

$$\begin{aligned}D\sum_{j=1}^{m_i}a_{ij}f_{ij}(z)&=\sum_{j=1}^{m_i}a_{ij}\frac{\partial f_{ij}}{\partial z}(z)-\sum_{j=1}^{m_i}\biggl(\sum_{l=1}^{m_i}Q_{lj}^{(i)}(z)a_{il}\biggr)f_{ij}(z)\\&=\sum_{l=1}^{m_i}a_{il}\frac{\partial f_{il}}{\partial z}(z)-\sum_{l=1}^{m_i}a_{il}\sum_{j=1}^{m_i}Q_{lj}^{(i)}(z)f_{ij}(z)\\&=\sum_{l=1}^{m_i}a_{il}\biggl(f'_{il}(z)-\sum_{j=1}^{m_i}Q_{lj}^{(i)}(z)f_{ij}(z)\biggr)=0,\qquad i=1,\dots,m.\end{aligned}\tag{2.7}$$

Hence if we apply D to functional forms (2.3) and multiply the result by $T(z)$, then we obtain forms of the same kind (with some other polynomial coefficients $P_{\bar\kappa}(z)$, $\bar\kappa\in\Omega$).

We now set

$$\begin{aligned}P_{\bar\kappa}^{[0]}(z)=&P_{\bar\kappa}(z),\qquad \bar\kappa\in\Omega,\\R_{\bar s}^{[0]}(z)=&R_{\bar s}(z),\qquad \bar s\in\Theta,\end{aligned}$$

where $P_{\bar\kappa}(z)$, $\bar\kappa\in\Omega$, and $R_{\bar s}(z)$, $\bar s\in\Theta$, are the polynomials constructed in Lemma 2.1 and the corresponding functions (2.6). (That is,

$$R^{[0]}(z;\bar a)=\sum_{\bar s\in\Theta}\bar a^{\bar s}R_{\bar s}^{[0]}(z).)$$

Then the functional forms

$$R^{[n]}(z;\bar{a}) = \big(T(z)D\big)^n R^{[0]}(z;\bar{a}), \qquad n \geqslant 0,$$

can be written as follows:

$$R^{[n]}(z;\bar{a}) = \sum_{\bar{u}\in U} P_{\bar{u}}^{[n]}(z;\bar{a}) \prod_{i=1}^{m} \big(a_{i1}f_{i1}(z) + \dots + a_{im_i}f_{im_i}(z)\big)^{N-u_i} \qquad n \geqslant 0,$$

and their coefficients are polynomials in z satisfying the recursion relations

$$P_{\bar{\kappa}}^{[n+1]}(z) = T(z)\biggl(\frac{d}{dz}P_{\bar{\kappa}}^{[n]}(z) - \sum_{i=1}^{m}\sum_{l,j=1}^{m_i}(\kappa_{ij} - \delta_{lj} + 1)Q_{lj}^{(i)}(z)P_{\bar{\kappa}-\bar{e}_{il}+\bar{e}_{ij}}^{[n]}(z)\biggr), \quad n \geqslant 0, \quad \bar{\kappa}\in\Omega. \tag{2.8}$$

The same relations holds also for the polynomials $R_{\bar{s}}^{[n]}(z)$ in the functions (0.9) ($\bar{s}\in\Theta$, $n \geqslant 0$) associated with each functional form

$$R^{[n]}(z;\bar{a}) = \sum_{\bar{s}\in\Theta} \bar{a}^{\bar{s}} R_{\bar{s}}^{[n]}(z), \qquad n \geqslant 0.$$

Namely,

$$R_{\bar{s}}^{[n+1]}(z) = T(z)\biggl(\frac{d}{dz}R_{\bar{s}}^{[n]}(z) - \sum_{i=1}^{m}\sum_{l,j=1}^{m_i}(s_{ij} - \delta_{lj} + 1)Q_{lj}^{(i)}(z)R_{\bar{s}-\bar{e}_{il}+\bar{e}_{ij}}^{[n]}(z)\biggr), \quad n \geqslant 0, \quad \bar{s}\in\Theta. \tag{2.9}$$

Now, setting

$$t = \max\Bigl\{\deg T, \max_{i,l,j}\{\deg TQ_{lj}^{(i)}\}\Bigr\},$$

we see from Lemma 2.1 and relations (2.8) and (2.9) that

$$\deg P_{\bar{\kappa}}^{[n]} < M + tn, \qquad n \geqslant 0, \quad \bar{\kappa}\in\Omega, \tag{2.10}$$

$$\operatorname*{ord}_{z=0} P_{\bar{\kappa}}^{[n]} \geqslant M - M_{\bar{u}} - n, \qquad n \geqslant 0, \quad \bar{\kappa}\in\Omega_{\bar{u}}, \quad \bar{u}\in U, \tag{2.11}$$

$$\operatorname*{ord}_{z=0} R_{\bar{s}}^{[n]} \geqslant K - n, \qquad n \geqslant 0, \quad \bar{s}\in\Theta. \tag{2.12}$$

Lemma 2.2. (a) *The normal expansions of the* $P_{\bar{\kappa}}^{[n]}(z)$, $n \geqslant 0$, $\bar{\kappa}\in\Omega$, *have integer coefficients*; *moreover,*

$$\max_{\bar{\kappa}\in\Omega}\bigl\{\|P_{\bar{\kappa}}^{[n]}(z)\|\bigr\} \leqslant (C_1N)^n\frac{(M+tn-1)!}{(M-1)!}\cdot C_0^{M/\varepsilon}, \qquad n \geqslant 0, \tag{2.13}$$

and therefore

$$\max_{\bar{\kappa}\in\Omega}\bigl\{\|P_{\bar{\kappa}}^{[n]}(z)\|\bigr\} < M^{C_3\varepsilon M} \tag{2.14}$$

for $n < C_2\varepsilon M$.

(b) *For* $n < C_2\varepsilon M$,

$$\left|R_{\bar{s}^*}^{[n]}(\alpha)\right| < M^{C_4\varepsilon M-K}, \qquad \bar{s}^* = N(\bar{e}_{11}+\bar{e}_{21}+\cdots+\bar{e}_{m1}). \tag{2.15}$$

Proof. (a) The first part of this assertion is a consequence of (2.8) and our choice of $T(z)$ (see (2.1)). We now set

$$C_5 = \left(1+\sum_{i=1}^{m} m_i^2\right)\cdot\max\Bigl\{\|T\|, \max_{i,l,j}\{\|TQ_{lj}^{(i)}\|\}\Bigr\}$$

and use the recursion relations (2.8) and the inequalities in Lemma 1.5 to obtain

$$\max_{\bar{\kappa}\in\Omega}\bigl\{\|P_{\bar{\kappa}}^{[n+1]}(z)\|\bigr\} \leqslant \binom{M+t(n+1)-1}{t}\cdot C_5N\cdot\max_{\bar{\kappa}\in\Omega}\bigl\{\|P_{\bar{\kappa}}^{[n]}(z)\|\bigr\}, \qquad n\geqslant 0.$$

Using now mere induction on n we obtain

$$\begin{aligned}\max_{\bar{\kappa}\in\Omega}\bigl\{\|P_{\bar{\kappa}}^{[n]}(z)\|\bigr\} &\leqslant \left(\frac{C_5N}{t!}\right)^n\frac{(M+tn-1)!}{(M-1)!}\cdot\max_{\bar{\kappa}\in\Omega}\bigl\{\|P_{\bar{\kappa}}^{[0]}(z)\|\bigr\}\\ &\leqslant \left(\frac{C_5N}{t!}\right)^n\frac{(M+tn-1)!}{(M-1)!}\cdot C_0^{M/\varepsilon}, \qquad n\geqslant 0,\end{aligned}$$

so that (2.13) holds. We can deduce inequalities (2.14) from (2.13) using the relations

$$\frac{(M+tn-1)!}{(M-1)!}\cdot N^n < (2M)^{tn}\cdot M^n, \qquad C_0^{M/\varepsilon} = o(M^{\varepsilon M}) \quad \text{as } M\to\infty$$

and $n < C_2\varepsilon M$.

(b) Let $P_{\bar{\kappa},\nu}^{[n]}$ ($\nu\in\mathbb{Z}^+$, $n\geqslant 0$, $\bar{\kappa}\in\Omega$) be the coefficients in the normal expansions of the polynomials $P_{\bar{\kappa}}^{[n]}(z)$, respectively; let $R_{\bar{s}^*,\mu}^{[n]}$ ($\mu\in\mathbb{Z}^+$,, $n\geqslant 0$) be the coefficients in the normal expansions of the forms $R_{\bar{s}^*}^{[n]}(z)$, and let $F_{\bar{u},\nu}$ ($\nu\in\mathbb{Z}^+$, $\bar{u}\in U$) be the coefficients in the normal expansions of the functions (0.11). Then by (2.6),

$$R_{\bar{s}^*}^{[n]}(z) = \sum_{\bar{u}\in U} P_{\bar{s}^*-u_1\bar{e}_{11}-u_2\bar{e}_{21}-\cdots-u_m\bar{e}_{m1}}^{[n]}(z)F_{\bar{u}}(z), \qquad n\geqslant 0,$$

therefore

$$R_{\bar{s}^*,\mu}^{[n]} = \sum_{\bar{u}\in U}\sum_{\nu=0}^{\mu}\binom{\mu}{\nu}P_{\bar{s}^*-u_1\bar{e}_{11}-u_2\bar{e}_{21}-\cdots-u_m\bar{e}_{m1},\nu}^{[n]}F_{\bar{u},\mu-\nu}, \qquad \mu\in\mathbb{Z}^+, \quad n\geqslant 0. \tag{2.16}$$

In addition,

$$R_{\bar{s}^*,\mu}^{[n]} = 0, \qquad \mu < K-n \quad \text{for } n < C_2\varepsilon M.$$

Hence using in (2.16) the estimates (2.14) and the definition of an E-function as applied to the collection involved in (0.11) we obtain (for $\mu \geqslant K-n$)

$$|R^{[n]}_{\bar{s}^*,\mu}| \leqslant \operatorname{Card} U \cdot \sum_{\nu=0}^{\mu} \binom{\mu}{\nu} \max_{\bar{\kappa}\in\Omega} \{\|P^{[n]}_{\bar{\kappa}}(z)\|\} C^{\mu+1} < C_6^{\mu+1} M^{C_3\varepsilon M}$$

and therefore

$$|R^{[n]}_{\bar{s}^*}(\alpha)| = \biggl| \sum_{\mu\geqslant K-n} \frac{R^{[n]}_{\bar{s}^*,\mu}}{\mu!}\alpha^{\mu} \biggr| < M^{C_3\varepsilon M} \sum_{\mu\geqslant K-n} \frac{|\alpha|^{\mu}}{\mu!} C_6^{\mu+1}.$$

By the inequality

$$\begin{aligned}
\sum_{\mu\geqslant K-n} \frac{|\alpha|^{\mu}}{\mu!} C_6^{\mu+1} &\leqslant \frac{|\alpha|^{K-n} C_6^{K-n+1}}{(K-n)!} \sum_{\mu\geqslant K-n} \frac{\bigl(C_6|\alpha|\bigr)^{\mu-K+n}}{(\mu-K+n)!} \\
&= \frac{|\alpha|^{K-n} C_6^{K-n+1}}{(K-n)!} e^{C_6|\alpha|} \\
&< \bigl(C_6|\alpha|\bigr)^{\operatorname{Card} U\cdot M} \left(\frac{e}{K-n}\right)^{K-n} e^{C_6|\alpha|} \\
&< \bigl(C_6|\alpha|\bigr)^{\operatorname{Card} U\cdot M} \left(\frac{e}{M}\right)^{K} e^{C_6|\alpha|} \\
&< e^{C_6|\alpha|} \bigl(C_6|\alpha| e\bigr)^{\operatorname{Card} U\cdot M} M^{-K}
\end{aligned}$$

and the relation $M < K-n \leqslant K < \operatorname{Card} U \cdot M$ we obtain

$$|R^{[n]}_{\bar{s}^*}(\alpha)| < M^{C_3\varepsilon M} e^{C_6|\alpha|} (C_6|\alpha| e)^{\operatorname{Card} U\cdot M} M^{-K}.$$

Since

$$e^{C_6|\alpha|} (C_6|\alpha| e)^{\operatorname{Card} U\cdot M} = o(M^{\varepsilon M}) \quad \text{as} \quad M\to\infty,$$

inequality (2.15) follows from the last estimate.

§ 3. Numerical approximating forms

In this section, as traditional in the Siegel–Shidlovskiĭ method, we proceed from the functional forms just constructed to numerical approximating forms. However, we shall build our arguments using a new scheme.

Ranks of numerical approximating forms. It is convenient to associate with each $\bar{s}\in\Theta$ a function $\mathcal{I}_{\bar{s}}\colon U\to\Omega$ defined as follows:

$$\mathcal{I}_{\bar{s}}(\bar{u}) = \bar{s} - u_1\bar{e}_{11} - u_2\bar{e}_{21} - \dots - u_m\bar{e}_{m1}, \qquad \bar{u} = (u_1, u_2, \dots, u_m) \in U.$$

Here $\mathcal{I}_{\bar{s}}$ is not necessarily defined for all $\bar{u}\in U$, because the components of the multi-index on the right-hand side are not necessarily non-negative.

Using this notation we can express the polynomials $R^{[n]}_{\bar{s}^*}(z)$ (here $n \geqslant 0$ and $\bar{s}^* = N(\bar{e}_{11} + \bar{e}_{21} + \dots + \bar{e}_{m1}))$ in the functions $f_{11}(z), f_{21}(z), \dots, f_{m1}(z)$ as follows:

$$R^{[n]}_{\bar{s}^*}(z) = \sum_{\bar{u}\in U} P^{[n]}_{\mathfrak{I}_{\bar{s}^*}(\bar{u})}(z) F_{\bar{u}}(z), \qquad n \geqslant 0. \tag{3.1}$$

We now set

$$\Omega^* = \bigcup_{\bar{u}\in U} \{\mathfrak{I}_{\bar{s}^*}(\bar{u})\}, \qquad \omega^* = \operatorname{Card}\Omega^* = \operatorname{Card} U.$$

An important point in the Siegel–Shidlovskiĭ method is the proof of the fact that the functional determinant made up of the forms under consideration (in our case this is $\det\bigl(P^{[n]}_{\bar{\kappa}}(z)\bigr)_{n=0,1,\dots,\omega-1;\bar{\kappa}\in\Omega})$ is non-zero. This is the kind of result Chudnovsky [7] proved using the condition that the collection of systems (0.6) be normal in the Siegel sense. As a matter of fact, we shall need only the forms (3.1) in numerical applications and therefore we can use a weaker version of that result, with less stringent (than the condition of normality) constraints. (On this account, see our remark to Theorem III).

Proposition 3.1. *Assume that a system of linear homogeneous differential equations* (0.5) *is in* $\mathbf{W}^0(\alpha)$, *the functions* (0.11) *are linearly independent over* $\mathbb{C}(z)$, *and* M *is sufficiently large* ($M > M'$). *Then the rank of the numerical matrix*

$$\bigl(P^{[n]}_{\bar{\kappa}}(\alpha)\bigr)_{n=0,1,\dots,\omega+[C_7\varepsilon M];\bar{\kappa}\in\Omega^*}$$

is precisely equal to ω^*.

Proof. Let $\widetilde{\omega}$, where $\widetilde{\omega} \leqslant \omega$, be the rank of the collection of linear forms $R^{[n]}(z;\bar{a})$, $n = 0, 1, 2, \dots$. Then $\widetilde{\omega} \geqslant 1$ because $R^{[0]}(z;\bar{a}) \not\equiv 0$ (see Remark 2 to Lemma 2.1).

For an arbitrary solution

$$a_{ij} = a_{ij}(z), \qquad j = 1, \dots, m_i, \quad i = 1, \dots, m,$$

of (0.6) the collection of functions

$$x_{\bar{\kappa}}(z) = \bar{a}^{\bar{\kappa}}(z) \prod_{i=1}^{m} \bigl(a_{i1}(z) f_{i1}(z) + \dots + a_{im_i}(z) f_{im_i}(z)\bigr)^{N-u_i}, \qquad \bar{\kappa} \in \Omega_{\bar{u}}, \quad \bar{u} \in U,$$

makes up a solution of the system of linear homogeneous differential equations

$$\frac{d}{dz} x_{\bar{\kappa}} = -\sum_{i=1}^{m} \sum_{l,j=1}^{m_i} \kappa_{ij} Q^{(i)}_{lj}(z) x_{\bar{\kappa}-\bar{e}_{ij}+\bar{e}_{il}}, \qquad \bar{\kappa} \in \Omega, \tag{3.2}$$

of order ω. By Lemma 7 in [1], Chapter 3, § 4 there exists a fundamental matrix of solutions $\bigl(x_{\bar{\kappa},\eta}(z)\bigr)_{\bar{\kappa}\in\Omega;\eta=1,\dots,\omega}$ of (3.2) such that setting

$$R^{[n]}_{\eta}(z) = \sum_{\bar{\kappa}\in\Omega} P^{[n]}_{\bar{\kappa}}(z) x_{\bar{\kappa},\eta}(z), \qquad n \geqslant 0, \quad \eta = 1, \dots, \omega$$

we obtain

$$R^{[n]}_{\eta}(z) \equiv 0, \qquad n \geqslant 0, \quad \eta = \widetilde{\omega} + 1, \dots, \omega. \tag{3.3}$$

Using the notation of § 1 we now set

$$\Lambda(z) = \det(x_{\bar{\kappa},\eta})_{\bar{\kappa}\in\Omega;\eta=1,\dots,\omega} \quad \text{and}$$

$$\lambda(\widetilde{\Omega}; z) = \det(x_{\bar{\kappa}',\eta})_{\bar{\kappa}'\in\Omega\setminus\widetilde{\Omega};\eta=\widetilde{\omega}+1,\dots,\omega}, \qquad \widetilde{\Omega} \subset \Omega, \quad \operatorname{Card}\widetilde{\Omega} = \widetilde{\omega}$$

(we set $\lambda(\Omega; z) = 1$ for $\widetilde{\omega} = \omega$).

Lemma 3.2. *For $M > M'$ there exists a set $\widetilde{\Omega} \subset \Omega$ containing Ω^* such that* $\operatorname{Card} \widetilde{\Omega} = \widetilde{\omega}$ *and* $\lambda(\widetilde{\Omega}; \alpha) \neq 0$.

Proof. If $\widetilde{\omega} = \omega$, then we can set $\widetilde{\Omega} = \Omega$. Hence only the case of $\widetilde{\omega} < \omega$ is worth considering here.

Assume that the lemma fails and $\lambda(\widetilde{\Omega}; \alpha) = 0$ for each $\widetilde{\Omega} \subset \Omega$ such that $\widetilde{\Omega} \supset \Omega^*$ and $\operatorname{Card} \widetilde{\Omega} = \widetilde{\omega}$. Hence the rank of the numerical matrix

$$\bigl(x_{\bar{\kappa},\eta}(\alpha)\bigr)_{\bar{\kappa}\in\Omega\setminus\Omega^*;\eta=\widetilde{\omega}+1,\dots,\omega}$$

is smaller than $\omega - \widetilde{\omega}$ and there exists a non-trivial linear combination (with numerical coefficients) $\bigl(x^*_{\bar{\kappa}}(z)\bigr)_{\bar{\kappa}\in\Omega}$ of the columns of the matrix

$$\bigl(x_{\bar{\kappa},\eta}(z)\bigr)_{\bar{\kappa}\in\Omega;\eta=\widetilde{\omega}+1,\dots,\omega}$$

such that

$$x^*_{\bar{\kappa}}(\alpha) = 0, \qquad \bar{\kappa} \in \Omega \setminus \Omega^*. \tag{3.4}$$

Thus, the column $\bigl(x^*_{\bar{\kappa}}(z)\bigr)_{\bar{\kappa}\in\Omega}$ is a non-trivial solution of (3.2), and

$$\sum_{\bar{\kappa}\in\Omega} P^{[n]}_{\bar{\kappa}}(z) x^*_{\bar{\kappa}}(z) \equiv 0, \qquad n \geqslant 0 \tag{3.5}$$

by (3.3).

The space of solutions of (3.2) satisfying (3.4) has dimension ω^*. The space spanned by the solutions

$$x_{\bar{\kappa}}(z) = A_{\bar{u}} \bar{\varphi}^{\bar{\kappa}}(z), \qquad A_{\bar{u}} \in \mathbb{C}, \quad \bar{\kappa} \in \Omega_{\bar{u}}, \quad \bar{u} \in U, \tag{3.6}$$

of (3.2), where the functions φ_{ij} (see (0.7)) satisfy (0.6) and the conditions (0.8), has the same dimension. At the same time, all solutions of the form (3.6) satisfy (3.4). Hence the converse is also true and the solution $\bigl(x^*_{\bar{\kappa}}(z)\bigr)_{\bar{\kappa}\in\Omega}$ just obtained can be represented as follows:

$$x^*_{\bar{\kappa}}(z) = A_{\bar{u}} \bar{\varphi}^{\bar{\kappa}}(z), \qquad \bar{\kappa} \in \Omega_{\bar{u}}, \quad \bar{u} \in U, \tag{3.7}$$

where $A_{\bar{u}} \in \mathbb{C}$ are certain constants.

Since the solution (3.7) of the system (3.2) is non-trivial, $A_{\bar{u}'} \neq 0$ for some $\bar{u}' \in U$. On the other hand,

$$\sum_{\bar{u}\in U} A_{\bar{u}} \sum_{\bar{\kappa}\in\Omega_{\bar{u}}} P^{[n]}_{\bar{\kappa}}(z) \bar{\varphi}^{\bar{\kappa}}(z) \equiv 0, \qquad n \geqslant 0$$

by (3.5), therefore

$$P^{[n]}_{\bar{\kappa}}(z) \equiv 0, \qquad n \geqslant 0, \quad \bar{\kappa} \in \Omega_{\bar{u}'}, \tag{3.8}$$

because the functions (0.7) are homogeneously algebraically independent over $\mathbb{C}(z)$.

Since $\widetilde{\omega} \geqslant 1$, there exist multi-indices $\bar{s} \in \Theta$ such that the polynomials

$$P^{[n]}_{\bar{s}-\bar{r}}(z), \qquad n \geqslant 0, \quad |\bar{r}| = u_i, \quad i = 1, \dots, m,$$

are not all trivial. In the set of such $\bar{s}$ we choose a multi-index $\bar{s}'$ with the largest sum $s'_{11} + s'_{21} + \dots + s'_{m1}$. Then each form

$$\begin{aligned} R^{[n]}_{\bar{s}'}(z) &= \sum_{\bar{u} \in U} \sum_{\substack{\bar{r}=(\bar{r}_1,\dots,\bar{r}_m) \\ |\bar{r}_i|=u_i, i=1,\dots,m}} \prod_{i=1}^{m} \frac{u_i!}{r_{i1}! \cdots r_{im_i}!} P^{[n]}_{\bar{s}'-\bar{r}}(z) \bar{f}^{\bar{r}}(z) \\ &= \sum_{\bar{u} \in U} P^{[n]}_{\mathcal{I}_{\bar{s}'}(\bar{u})}(z) f_{11}^{u_1}(z) f_{21}^{u_2}(z) \cdots f_{m1}^{u_m}(z), \qquad n \geqslant 0, \end{aligned} \tag{3.9}$$

involves at most ω^* polynomials. Let $\widetilde{\omega}'$ be the rank of the collection of the forms (3.9) over $\mathbb{C}(z)$. Then $\widetilde{\omega}' \geqslant 1$ by our choice of $\bar{s}'$ and $\widetilde{\omega}' < \omega^*$ by (3.8). Hence there exists a non-empty subset

$$\widetilde{\Omega}' \subset \Omega' = \bigcup_{\bar{u} \in U} \{\mathcal{I}_{\bar{s}'}(\bar{u})\}, \qquad \operatorname{Card} \widetilde{\Omega}' = \widetilde{\omega}',$$

and rational functions

$$D_{\bar{\kappa},\bar{\kappa}'}(z), \qquad \bar{\kappa} \in \widetilde{\Omega}', \quad \bar{\kappa}' \in \Omega' \setminus \widetilde{\Omega}',$$

such that

$$P^{[n]}_{\bar{\kappa}'}(z) = \sum_{\bar{\kappa} \in \widetilde{\Omega}'} P^{[n]}_{\bar{\kappa}}(z) D_{\bar{\kappa},\bar{\kappa}'}(z), \qquad n \geqslant 0, \quad \bar{\kappa}' \in \Omega' \setminus \widetilde{\Omega}'. \tag{3.10}$$

Hence there exists a set $\widetilde{\Omega} \subset \Omega$ containing $\widetilde{\Omega}'$ such that $\operatorname{Card} \widetilde{\Omega} = \widetilde{\omega}$ and

$$P^{[n]}_{\bar{\kappa}'}(z) = \sum_{\bar{\kappa} \in \widetilde{\Omega}} P^{[n]}_{\bar{\kappa}}(z) D_{\bar{\kappa},\bar{\kappa}'}(z), \qquad n \geqslant 0, \quad \bar{\kappa}' \in \Omega \setminus \widetilde{\Omega},$$

fore some rational functions

$$D_{\bar{\kappa},\bar{\kappa}'}(z), \qquad \bar{\kappa} \in \widetilde{\Omega}, \quad \bar{\kappa}' \in \Omega \setminus \widetilde{\Omega}, \tag{3.11}$$

where

$$D_{\bar{\kappa},\bar{\kappa}'} = 0, \qquad \bar{\kappa} \in \widetilde{\Omega} \setminus \widetilde{\Omega}', \quad \bar{\kappa}' \in \Omega' \setminus \widetilde{\Omega}'.$$

Reasoning as in [8], § 3, proof of Lemma 3.3 and using an analogue of Lemma 3.1 and Lemma 3.2 we can show that the rational functions (3.11) can be represented as follows:

$$\begin{gathered} D_{\bar{\kappa},\bar{\kappa}'}(z) = \frac{B_{\bar{\kappa},\bar{\kappa}'}(z)}{B(z)}, \quad \text{where} \quad B, B_{\bar{\kappa},\bar{\kappa}'} \in \mathbb{C}[z], \quad B \not\equiv 0, \\ \deg B \leqslant C_8 \omega N, \quad \deg B_{\bar{\kappa},\bar{\kappa}'} \leqslant C_8 \omega N, \qquad \bar{\kappa} \in \widetilde{\Omega}, \quad \bar{\kappa}' \in \Omega \setminus \widetilde{\Omega}, \end{gathered} \tag{3.12}$$

and C_8 depends only on the collections of functions (0.9).

Hence there exists a set $\mathcal{N} \subset \{0, 1, \dots, \widetilde{\omega} - 1\}$ such that $\operatorname{Card} \mathcal{N} = \widetilde{\omega}'$ and

$$\Delta(z) = \det\bigl(P_{\bar\kappa}^{[n]}(z)\bigr)_{n \in \mathcal{N}; \bar\kappa \in \widetilde{\Omega}'} \not\equiv 0.$$

By estimates (2.10) we now obtain

$$\deg \Delta < \widetilde{\omega}' M + \omega^2 t. \tag{3.13}$$

From the representation (3.12) and equalities (3.10) we can see that

$$\begin{aligned} B(z) R_{\bar s'}^{[n]}(z) &= \sum_{\bar\kappa \in \widetilde{\Omega}'} P_{\bar\kappa}^{[n]}(z) B(z) \bar f^{\bar s' - \bar\kappa}(z) + \sum_{\bar\kappa' \in \Omega' \setminus \widetilde{\Omega}'} P_{\bar\kappa'}^{[n]}(z) B(z) \bar f^{\bar s' - \bar\kappa'}(z) \\ &= \sum_{\bar\kappa \in \widetilde{\Omega}'} P_{\bar\kappa}^{[n]}(z) \biggl(B(z) \bar f^{\bar s' - \bar\kappa}(z) + \sum_{\bar\kappa' \in \Omega' \setminus \widetilde{\Omega}'} B_{\bar\kappa, \bar\kappa'}(z) \bar f^{\bar s' - \bar\kappa'}(z) \biggr) \\ &= \sum_{\bar\kappa \in \widetilde{\Omega}'} P_{\bar\kappa}^{[n]}(z) \widetilde{x}_{\bar\kappa}(z), \qquad n \in \mathcal{N}, \end{aligned} \tag{3.14}$$

where the functions $\widetilde{x}_{\bar\kappa} \in \mathbb{C}[z, f_{11}, f_{21}, \dots, f_{m1}]$, $\bar\kappa \in \widetilde{\Omega}'$, are of degree at most $C_8 \omega N$ in z and of degree d with respect to the collection $f_{11}, f_{21}, \dots, f_{m1}$. In addition, $\widetilde{x}_{\bar\kappa} \not\equiv 0$ for $\bar\kappa \in \widetilde{\Omega}'$ because $B(z) \not\equiv 0$ and the functions (0.11) involved in the definitions of the $\widetilde{x}_{\bar\kappa}$, $\bar\kappa \in \widetilde{\Omega}'$, are linearly independent over $\mathbb{C}(z)$. By Theorem 1 in [9] (and remarks to this theorem in the case of algebraically dependent functions (0.9)), the order of the zero at $z = 0$ of each of the $\widetilde{x}_{\bar\kappa}(z)$ is at most

$$C_9 (C_8 \omega N + 1) d^{m_1 + \dots + m_m} < C_{10} \omega N.$$

By construction, $\widetilde{\Omega}'$ intersects each of the $\Omega_{\bar u}$, $\bar u \in U$, by at most one element. We set

$$U' = \{\bar u : \widetilde{\Omega}' \cap \Omega_{\bar u} \neq \varnothing\} \subset U; \quad \text{then} \quad \operatorname{Card} U' = \widetilde{\omega}'.$$

Let $\bar u^* \in U$ be such that $M = M_{\bar u^*}$. We set $\bar r = \widetilde{\Omega}' \cap \Omega_{\bar u^*}$ for $\bar u^* \in U'$, otherwise let $\bar r$ be an element of $\widetilde{\Omega}'$. In both cases the set

$$U'' = \{\bar u : \widetilde{\Omega}' \setminus \{\bar r\} \cap \Omega_{\bar u} \neq \varnothing\} \subset U', \quad \text{where } \operatorname{Card} U'' = \operatorname{Card} U' - 1 = \widetilde{\omega}' - 1,$$

does not contain $\bar u^*$.

We now multiply the matrix

$$\bigl(P_{\bar\kappa}^{[n]}(z)\bigr)_{n \in \mathcal{N}; \bar\kappa \in \widetilde{\Omega}'}$$

with determinant $\Delta(z)$ by the matrix

$$\bigl(\widetilde{x}_{\bar\kappa}(z) \mid \delta_{\bar\kappa, \bar\kappa'}\bigr)_{\bar\kappa \in \widetilde{\Omega}'; \bar\kappa' \in \widetilde{\Omega}' \setminus \{\bar r\}},$$

with determinant $\widetilde{x}_{\bar{r}}$ on the right. The resulting matrix

$$\big(B(z)R_{\bar{s}'}^{[n]}(z) \mid P_{\bar{\kappa}'}^{[n]}(z)\big)_{n\in\mathcal{N};\bar{\kappa}'\in\widetilde{\Omega}'\setminus\{\bar{r}\}}$$

has the non-zero determinant $\Delta(z)\widetilde{x}_{\bar{r}}(z)$ by (3.14).

The components of the first column of this matrix are functional forms with zero orders at $z=0$ at least $K-\widetilde{\omega}$ in view of (2.12); by (2.11), the zero orders at $z=0$ of the polynomials $P_{\bar{\kappa}'}^{[n]}(z)$, $n\in\mathcal{N}$, are at least $M-M_{\bar{u}}-\widetilde{\omega}$ for $\bar{\kappa}'\in\Omega_{\bar{u}}$, $\bar{\kappa}'\in\widetilde{\Omega}'\setminus\{\bar{r}\}$. Hence

$$\operatorname*{ord}_{z=0}\Delta\widetilde{x}_{\bar{r}}\geqslant K+\sum_{\bar{u}\in U''}(M-M_{\bar{u}})-\widetilde{\omega}'\cdot\widetilde{\omega}>K+\sum_{\bar{u}\in U''}(M-M_{\bar{u}})-\omega^2,$$

therefore

$$\begin{aligned}\operatorname*{ord}_{z=0}\Delta &> K+\sum_{\bar{u}\in U''}(M-M_{\bar{u}})-\omega^2-\operatorname*{ord}_{z=0}\widetilde{x}_{\bar{r}}\\ &> K+\sum_{\bar{u}\in U''}(M-M_{\bar{u}})-\omega^2-C_{10}\omega N\\ &\geqslant \operatorname{Card}U''\cdot M+\sum_{\bar{u}\in U\setminus U''}M_{\bar{u}}-2\varepsilon M-\omega^2-C_{10}\omega N\\ &= \widetilde{\omega}'M+\sum_{\substack{\bar{u}\in U\setminus U''\\ \bar{u}\neq\bar{u}^*}}M_{\bar{u}}-2\varepsilon M-\omega^2-C_{10}\omega N.\end{aligned}$$

Comparing the last estimate and (3.13) we see that

$$\sum_{\substack{\bar{u}\in U\setminus U''\\ \bar{u}\neq\bar{u}^*}}M_{\bar{u}}<2\varepsilon M+\omega^2(t+1)+C_{10}\omega N<3\varepsilon M$$

for all $M>M'$. Here the sum on the left-hand side is taken over a non-empty set because $\operatorname{Card}U''=\widetilde{\omega}'-1\leqslant\omega^*-2=\operatorname{Card}U-2$. On the other hand $M_{\bar{u}}\geqslant 3\varepsilon M$ for all $\bar{u}\in U$ by the choice of the parameters in our construction, which contradicts the last inequality.

To sum up, the initial assumption fails and therefore the assertion of Lemma 3.2 holds.

If $\widetilde{\Omega}\subset\Omega$, $\operatorname{Card}\widetilde{\Omega}=\widetilde{\omega}$, is the set that exists by Lemma 3.2, then we set $\Delta(z)=\det\big(P_{\bar{\kappa}}^{[n]}(z)\big)_{n=0,1,\dots,\widetilde{\omega}-1;\bar{\kappa}\in\widetilde{\Omega}}$. By Lemma 1.3 we have $\Delta(z)\not\equiv 0$. Moreover, if we replace some column in the matrix of the determinant $\Delta(z)$ by the column $\big(R_{\bar{s}^*}^{[n]}(z)\big)_{n=0,1,\dots,\widetilde{\omega}-1}$ using a non-degenerate linear transformation, then

$$\operatorname*{ord}_{z=0}\Delta(z)>K-\widetilde{\omega}\geqslant M \tag{3.15}$$

by (2.12). By the estimates (2.10), (2.13), and (3.15) in Lemma 1.6 we obtain

$$
\begin{aligned}
\operatorname*{ord}_{z=\alpha}\Delta(z) &< \frac{2\widetilde{\omega}}{\log M}\biggl(\log\Bigl((C_1N)^{\widetilde{\omega}}\frac{(M+t\widetilde{\omega}-1)!}{(M-1)!}C_0^{M/\varepsilon}\Bigr)+2(M+t\widetilde{\omega}-1)(1+\log\widetilde{\omega})\biggr)\\
&\leqslant \frac{2\omega}{\log M}\Bigl(\omega\log(C_1N)+t\omega\log(2M)+\frac{M}{\varepsilon}\log C_0+2(M+t\omega)(1+\log\omega)\Bigr)\\
&\leqslant \frac{C_{11}\omega M}{\varepsilon\log M}\leqslant C_7\varepsilon M \qquad (3.16)
\end{aligned}
$$

because

$$\frac{\omega}{\varepsilon\log M}\asymp\varepsilon$$

in view of our choice of the parameters (see (2.2)).

To complete the proof of Proposition 3.1 it remains to use Lemma 1.3 with estimate (3.16) and the fact that $\Omega^*\subset\widetilde{\Omega}$.

Arithmetic properties of numerical forms. In this subsection we sum up our results and prove Theorem II following Baker [3].

Proposition 3.3. *Assume that the system of linear homogeneous differential equations* (0.5) *is in* $\mathbf{W}^0(\alpha)$*, the functions* (0.11) *are linearly independent over* $\mathbb{C}(z)$*, and* $M>M'$*. Then there exist integers* $\rho_{\bar u}^{[n]}$*,* $n=1,\dots,\operatorname{Card}U$*,* $\bar u\in U$*, such that*

$$|\rho_{\bar u}^{[n]}|<M^{M_{\bar u}+C_{12}\varepsilon M}, \qquad (3.17)$$

the numerical forms

$$\xi^{[n]}=\sum_{\bar u\in U}\rho_{\bar u}^{[n]}F_{\bar u}(\alpha)=\sum_{\bar u\in U}\rho_{\bar u}^{[n]}f_{11}^{u_1}(\alpha)f_{21}^{u_2}(\alpha)\cdots f_{m1}^{u_m}(\alpha), \qquad n=1,\dots,\operatorname{Card}U, \qquad (3.18)$$

satisfy the estimates

$$|\xi^{[n]}|<M^{-\sum_{\bar u\in U}M_{\bar u}+M+C_{13}\varepsilon M}, \qquad n=1,\dots,\operatorname{Card}U, \qquad (3.19)$$

and, moreover,

$$\det\bigl(\rho_{\bar u}^{[n]}\bigr)_{n=1,\dots,\operatorname{Card}U;\bar u\in U}\neq 0. \qquad (3.20)$$

Proof. We use Proposition 3.1, according to which there exist integers $\nu_1,\dots,\nu_{\omega^*}$ (where $\omega^*=\operatorname{Card}U$) such that $0\leqslant\nu_1<\dots<\nu_{\omega^*}\leqslant\omega+[C_7\varepsilon M]<C_2\varepsilon M$ and

$$\det\bigl(P_{\bar\kappa}^{[\nu_n]}(\alpha)\bigr)_{n=1,\dots,\omega^*;\bar\kappa\in\Omega^*}\neq 0.$$

Let $\alpha=a/b$, where $a\in\mathbb{Z}$ and $b\in\mathbb{N}$. Then, in view of (2.14), the integers

$$\rho_{\bar u}^{[n]}=b^{M+t\nu_n}(M+t\nu_n)!P_{\mathcal{J}_{\bar s^*}(\bar u)}^{[n]}(\alpha), \qquad n=1,\dots,\omega^*, \quad \bar u\in U$$

satisfy the estimate

$$\begin{aligned}|\rho_{\bar u}^{[n]}| &< b^{M+t\nu_n}(M+t\nu_n)! \sum_{\mu=\max\{0,M-M_{\bar u}-\nu_n\}}^{M+t\nu_n-1} \frac{M^{C_3\varepsilon M}}{\mu!}|\alpha|^{\mu} \\ &< b^{M+tC_2\varepsilon M}|a|^M e^{|\alpha|}(2e)^M(2M)^{(t+1)C_2\varepsilon M}M^{M_{\bar u}} \\ &\leqslant M^{M_{\bar u}+C_{12}\varepsilon M}, \qquad n=1,\dots,\omega^*, \quad \bar u\in U,\end{aligned}$$

because

$$\begin{aligned}\sum_{\mu=\max\{0,M-M_{\bar u}-\nu\}}^{M+t\nu-1} \frac{M^{C_3\varepsilon M}}{\mu!}|\alpha|^{\mu} &< \frac{|\alpha|^{\max\{0,M-M_{\bar u}-\nu\}}}{\max\{0,M-M_{\bar u}-\nu\}!}\sum_{\mu=0}^{\infty}\frac{|\alpha|^{\mu}}{\mu!} \\ &\leqslant \frac{|a|^M}{\max\{0,M-M_{\bar u}-\nu\}!}\cdot e^{|\alpha|},\end{aligned}$$

$$\begin{aligned}\frac{(M+t\nu)!}{\max\{0,M-M_{\bar u}-\nu\}!} &< \frac{(M+t\nu+\nu)!}{(M-M_{\bar u})!} < \frac{(2M)^{M+t\nu+\nu}}{(M-M_{\bar u})^{M-M_{\bar u}}\cdot e^{-(M-M_{\bar u})}} \\ &< \frac{(2M)^{M+t\nu+\nu}}{M^{M-M_{\bar u}}\cdot e^{-M}} = (2e)^M(2M)^{(t+1)\nu}M^{M_{\bar u}},\end{aligned}$$

$$\nu < \frac{M}{t+1}, \qquad \bar u\in U.$$

(In the last estimate we used the inequality

$$\frac{M^{M-M_{\bar u}}}{(M-M_{\bar u})^{M-M_{\bar u}}} = \left(1+\frac{M_{\bar u}}{M-M_{\bar u}}\right)^{M-M_{\bar u}} < e^{M_{\bar u}}, \qquad \bar u\in U.)$$

We now apply the estimate (2.15) to the forms (3.18) under consideration to obtain

$$\begin{aligned}|\xi^{[n]}| &= b^{M+t\nu_n}(M+t\nu_n)!|R_{\bar s^*}^{[\nu_n]}(\alpha)| < b^{M+tC_2\varepsilon M}(2M)^{M+tC_2\varepsilon M}M^{C_4\varepsilon M-K} \\ &\leqslant M^{-\sum_{\bar u\in U}M_{\bar u}+M+C_{13}\varepsilon M}, \qquad n=1,\dots,\omega^*,\end{aligned}$$

which proves the proposition.

Proof of Theorem II. We set

$$C_{14} = \frac12\min_{\bar u\in U}\{|F_{\bar u}(\alpha)|\},$$

$$C_{15} = \max\left\{3, (\omega^*-2)C_{12}+C_{13}+\frac{1}{\varepsilon(M')\cdot M'\log M'}\,\log\frac{\omega^*!}{C_{14}}\right\},$$

where M', C_{12}, and C_{13} are the constants defined in Proposition 3.3, $\omega^* = \operatorname{Card} U$, and

$$\varepsilon = \varepsilon(M) \asymp (\log M)^{-1/(m_1+\cdots+m_m-m+2)} \tag{3.21}$$

is as defined in (2.2). Then for all $M > M'$ we have $C_{15} \geqslant 3$ and

$$\omega^*! M^{(\omega^*-2)C_{12}\varepsilon M + C_{13}\varepsilon M - C_{15}\varepsilon M} \leqslant C_{14}. \tag{3.22}$$

Given a numerical form

$$r = \sum_{\bar{u}\in U} h_{\bar{u}} F_{\bar{u}}(\alpha), \qquad h_{\bar{u}} \in \mathbb{Z}\setminus\{0\}, \quad \bar{u}\in U, \tag{3.23}$$

let M be the smallest integer such that

$$M^{(1-C_{15}\varepsilon)M} \geqslant H = \max_{\bar{u}\in U}\{|h_{\bar{u}}|\} \geqslant 3. \tag{3.24}$$

Then for all $M > M''$ we have $H > M^{M/2}$ and, in particular,

$$\log\log H > \log M + \log\log M - \log 2 > \log M,$$

therefore by (3.21),

$$\varkappa = \varkappa(H) = (\log\log H)^{-1/(m_1+\cdots+m_m-m+2)} > C_{16}\varepsilon.$$

Consequently, for $M > M''$ we have

$$M^{\varepsilon M} < H^{2\varepsilon} < H^{2\varkappa/C_{16}}. \tag{3.25}$$

We now choose $\bar{u}^* \in U$ such that $H = |h_{\bar{u}^*}|$ and we set $M_{\bar{u}^*} = M$. We choose the integers $M_{\bar{u}}$, $\bar{u} \neq \bar{u}^*$, so that

$$M_{\bar{u}} \geqslant \frac{\log|h_{\bar{u}}|}{\log M} + C_{15}\varepsilon M > M_{\bar{u}} - 1, \qquad \bar{u}\in U\setminus\{\bar{u}^*\}. \tag{3.26}$$

Then, in particular,

$$M_{\bar{u}} \geqslant \frac{\log|h_{\bar{u}}|}{\log M} + C_{15}\varepsilon M \geqslant C_{15}\varepsilon M \geqslant 3\varepsilon M, \qquad \bar{u}\in U\setminus\{\bar{u}^*\}. \tag{3.27}$$

By (3.24) and (3.26),

$$M \geqslant \frac{\log H}{\log M} + C_{15}\varepsilon M \geqslant \frac{\log|h_{\bar{u}}|}{\log M} + C_{15}\varepsilon M > M_{\bar{u}} - 1, \qquad \bar{u}\in U\setminus\{\bar{u}^*\},$$

therefore

$$M = M_{\bar{u}^*} = \max_{\bar{u}\in U}\{M_{\bar{u}}\}. \tag{3.28}$$

Conditions (3.27) and (3.28) ensure (2.2). In addition we can assume that $M > M'$ and $M > M''$ because there are only finitely many forms (3.23) such that $M \leqslant \max\{M', M''\}$, and therefore the estimate of the theorem holds for these forms. Thus, we can use Proposition 3.3. The forms (3.18) are linearly independent by (3.20). Hence we can choose $(m-1)$ of these forms, say, $\xi^{[2]},\dots,\xi^{[\omega^*]}$, such that $r,\xi^{[2]},\dots,\xi^{[\omega^*]}$ are linearly independent. We now consider the determinant

$$\tau=\det\begin{pmatrix} h_{\bar u}\\ \rho_{\bar u}^{[2]}\\ \vdots\\ \rho_{\bar u}^{[\omega^*]}\end{pmatrix}_{\bar u\in U}$$

of the matrix formed by the coefficients of these forms. It does not vanish because the forms (3.23) and (3.18) are linearly independent. Since $\tau\in\mathbb{Z}$, it follows that $|\tau|\geqslant 1$. We now consider the matrix corresponding to the determinant τ. We multiply the $\bar u^*$th column in this matrix by $F_{\bar u^*}(\alpha)$ and add to it the columns with indices $\bar u\in U\setminus\{\bar u^*\}$ multiplied by $F_{\bar u}(\alpha)$, respectively. The determinant of the resulting matrix is $\tau F_{\bar u^*}(\alpha)$; on the other hand we can represent it as

$$\tau_1 r+\sum_{n=2}^{\omega^*}\tau_n\xi^{[n]},$$

where τ_n is the algebraic complement of the entry in the nth row and the $\bar u^*$th column of τ. Consequently,

$$\begin{aligned}|\tau_1|\cdot|r|&=\biggl|\tau F_{\bar u^*}(\alpha)-\sum_{n=2}^{\omega^*}\tau_n\xi^{[n]}\biggr|\geqslant|\tau|\cdot|F_{\bar u^*}(\alpha)|-\sum_{n=2}^{\omega^*}|\tau_n|\cdot|\xi^{[n]}|\\ &\geqslant|F_{\bar u^*}(\alpha)|-\sum_{n=2}^{\omega^*}|\tau_n|\cdot|\xi^{[n]}|\geqslant 2C_{14}-\sum_{n=2}^{\omega^*}|\tau_n|\cdot|\xi^{[n]}|.\end{aligned}\tag{3.29}$$

From the estimates (3.17) and in view of the inequality

$$|h_{\bar u}|\leqslant M^{M_{\bar u}-C_{15}\varepsilon M},\qquad \bar u\in U\setminus\{\bar u^*\},$$

following from (3.26), we see that

$$\begin{aligned}&|\tau_1|<(\omega^*-1)!\prod_{\bar u\neq\bar u^*}M^{M_{\bar u}+C_{12}\varepsilon M}, &&\qquad (3.30)\\ &|\tau_n|<(\omega^*-1)!\prod_{\bar u\neq\bar u^*}M^{M_{\bar u}}\cdot M^{(\omega^*-2)C_{12}\varepsilon M-C_{15}\varepsilon M},\qquad n=2,\dots,\omega^*,\end{aligned}$$

therefore by (3.19) and (3.22) we obtain

$$|\tau_n|\cdot|\xi^{[n]}|<(\omega^*-1)!M^{(\omega^*-2)C_{12}\varepsilon M-C_{15}\varepsilon M+C_{13}\varepsilon M}\leqslant\frac{C_{14}}{\omega^*},\qquad n=2,\dots,\omega^*.$$

Substituting this inequality in (3.29) we obtain

$$|\tau_1|\cdot|r|>2C_{14}-(\omega^*-1)\frac{C_{14}}{\omega^*}>C_{14},$$

therefore from (3.25), (3.26), and (3.30) we see that

$$|r| > C_{14}|\tau_1|^{-1} > \frac{C_{14}}{(\omega^*-1)!}\prod_{\bar u\ne\bar u^*}\bigl(|h_{\bar u}|M^{C_{15}\varepsilon M+1+C_{12}\varepsilon M}\bigr)^{-1}$$

$$> \frac{C_{14}}{(\omega^*-1)!}\prod_{\bar u\ne\bar u^*}|h_{\bar u}|^{-1}\cdot H^{-2(\omega^*-1)(C_{15}+1+C_{12})\varkappa/C_{16}}.$$

The last inequality is just the estimate of Theorem II with $C = C_{14}/(\omega^*-1)!$ and $\gamma = 2(\omega^*-1)(C_{15}+1+C_{12})/C_{16}$.

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