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On the irrationality measure for a q-analogue of $\zeta(2)$

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Abstract. A Liouville-type estimate is proved for the irrationality measure of the quantities

$$\zeta_q(2) = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2}$$

with $q^{-1} \in \mathbb{Z} \setminus \{0, \pm 1\}$. The proof is based on the application of a q-analogue of the arithmetic method developed by Chudnovsky, Rukhadze, and Hata and of the transformation group for hypergeometric series — the group-structure approach introduced by Rhin and Viola.

Bibliography: 21 titles.

Introduction

For a complex number q, $|q| < 1$, we consider the quantities

$$\zeta_q(1) := \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} = \sum_{n=1}^{\infty} \sigma_0(n) q^n, \qquad \zeta_q(2) := \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} = \sum_{n=1}^{\infty} \sigma_1(n) q^n, \quad (1)$$

where $\sigma_k(n) = \sum_{l|n} l^k$ for $k = 0, 1$. Multiplying the series in (1) by $(1-q)^k$, $k = 1, 2$, and letting $q \to 1 - 0$ term by term we obtain the (divergent) harmonic series and the (convergent) series for $\zeta(2)$, respectively. The irrationality of the q-harmonic series $\zeta_q(1)$ with $q = 1/p$, where $p \in \mathbb{Z} \setminus \{0, \pm 1\}$, has been proved by Bézivin [1] and independently by Borwein [2]; the irrationality of $\zeta_q(2)$ for the same values of q has been established by Duverney [3]. However, the general result of Nesterenko [4] on arithmetic properties of the values of modular functions yields the transcendence of $\zeta_q(2)$ for arbitrary algebraic q, $0 < |q| < 1$. The aim of our work is to find the irrationality measure of $\zeta_q(2)$ for $q = 1/p$, $p \in \mathbb{Z} \setminus \{0, \pm 1\}$. Namely, we prove the following result.

Theorem. *Let $q = 1/p$, where $p \in \mathbb{Z} \setminus \{0, \pm 1\}$, and let*

$$\zeta_q(2) = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} = \sum_{n=1}^{\infty} \frac{p^n}{(p^n-1)^2}. \qquad (2)$$

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Then $\zeta_q(2)$ *is an irrational number and the inequality*

$$\left|\zeta_q(2)-\frac{a}{b}\right|\leqslant|b|^{-4.07869375}$$

has finitely many integer-valued solutions a *and* b*.*

We emphasize that Nesterenko's Theorem 2 in [5] yields the estimate

$$\left|\zeta_q(2)-\frac{a}{b}\right|>|b|^{-\gamma\ln^9\max\{2,\ln|b|\}}$$

for all $a,b\in\mathbb{Z}$, where γ depends only on $q\in\mathbb{Q}$, $0<|q|<1$. Thus, the theorem we prove here provides a qualitative improvement of the characteristics of irrationality of the quantities (2) in the case when $q^{-1}\in\mathbb{Z}\setminus\{0,\pm1\}$: it gives a Liouville-type estimate for the irrationality measure. Recall that the *irrationality exponent* of a real irrational number α is defined by the relation

$$\mu=\mu(\alpha):=\inf\{c\in\mathbb{R}: \text{the inequality } |\alpha-a/b|\leqslant|b|^{-c} \text{ has finitely many solutions in } a,b\in\mathbb{Z}\};$$

if $\mu(\alpha)<+\infty$, then α is said to be of *Liouville type*. In this notation the statement of the theorem can be written in the form of the following inequality:

$$\mu(\zeta_q(2))\leqslant 4.07869374\dots. \tag{3}$$

The proof of the theorem below is based on a q-analogue of the method proposed by Rhin and Viola for sharpening the estimate of the irrationality measure of $\zeta(2)=\pi^2/6$; namely, in [6] they prove the inequality $\mu(\zeta(2))\leqslant 5.44124250\dots$, which is the best result of this kind for $\zeta(2)$ known to date. The above-mentioned method, the group-structure approach, has been developed further to the record value $\mu(\zeta(3))\leqslant 5.51389062\dots$ for the Apéry constant; the last result is also due to Rhin and Viola [7]. In this paper we demonstrate the potentials of the group-structure approach for the solution of another number-theoretic problem: we adhere to the q-analogue of the general scheme of [6], [8], [9]. The main q-arithmetic ingredients for the application of the Rhin–Viola method are established in [10], [11]; we formulate them in § 1 to make our exposition self-contained. We prove the theorem in § 2–6 and present in § 7 a sequence of linear forms that is a q-analogue of Apéry's sequence [12] used for the proof of the irrationality of $\zeta(2)$; of course, the sequence in § 7 also ensures that $\zeta_q(2)$ is irrational of Liouville type for $q^{-1}\in\mathbb{Z}\setminus\{0,\pm1\}$. By these means we give an affirmative answer to Van Assche's question [13] on the proof of the irrationality of a q-extension of $\zeta(2)$ "in the spirit of Apéry", although we do not interpret Apéry's q-sequence in terms of difference equations and/or orthogonal polynomials.

§ 1. q-arithmetic

Recall the standard q-notation (see [14], Chapter 1):

$$(a;q)_n := \prod_{\nu=1}^{n}(1-aq^{\nu-1}), \qquad (a_1,a_2,\dots,a_m;q)_n := (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n,$$

$$\Gamma_q(t) := \frac{(q;q)_\infty}{(q^t;q)_\infty}(1-q)^{1-t}, \qquad [n]_q! := \Gamma_q(n+1) = \frac{(q;q)_n}{(1-q)^n},$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q!\,[n-k]_q!} = \frac{(q;q)_n}{(q;q)_k\cdot(q;q)_{n-k}},$$

where $k=0,1,\dots,n$ and $n=0,1,2,\dots$.

Consider the *cyclotomic polynomials*

$$\Phi_l(x) := \prod_{\substack{k=1\\(k,l)=1}}^{l}(x-e^{2\pi ik/l}), \quad \deg\Phi_l(x)=\varphi(l), \qquad l=1,2,\dots, \tag{4}$$

where $\varphi(\,\cdot\,)$ is the Euler totient function. As is well known, the coefficients of the polynomials (4) are integers ([15], Theorem 13.3) and for each $l=1,2,\dots$ the polynomial $\Phi_l(x)$ is irreducible over $\mathbb{Z}$ ([15], Theorem 13.4; see also [16], § 60); in addition, the formula

$$x^n-1=\prod_{l\mid n}\Phi_l(x) \tag{5}$$

holds. Since

$$(x;x)_n=(1-x)(1-x^2)\cdots(1-x^n)=\pm\prod_{k=1}^{n}\prod_{l\mid k}\Phi_l(x),$$

the factorization of $(x;x)_n$ in a product of irreducible polynomials contains only polynomials (4) and

$$\operatorname{ord}_{\Phi_l(x)}(x;x)_n=\left\lfloor\frac{n}{l}\right\rfloor, \qquad l=1,2,\dots, \tag{6}$$

where $\lfloor\,\cdot\,\rfloor$ is the integer part of a number. A simple consequence of (6) is the formula

$$\operatorname{ord}_{\Phi_l(x)}\begin{bmatrix} n \\ k \end{bmatrix}_x=\left\lfloor\frac{n}{l}\right\rfloor-\left\lfloor\frac{k}{l}\right\rfloor-\left\lfloor\frac{n-k}{l}\right\rfloor, \tag{7}$$

which makes it possible to regard cyclotomic polynomials as q-analogues of primes. Formula (7) also yields the inclusions

$$\begin{bmatrix} n \\ k \end{bmatrix}_x\in\mathbb{Z}[x], \qquad k=0,1,\dots,n, \quad n=0,1,2,\dots, \tag{8}$$

which are usually deduced from the q-version of the Pascal triangle.

The factorization (5) shows that the polynomial

$$D_n(x) := \prod_{l=1}^{n} \Phi_l(x) \in \mathbb{Z}[x]$$

is the least common multiple of the polynomials $x-1, x^2-1, \dots, x^n-1$ or, equivalently, $D_n(x)$ is the least-degree polynomial satisfying the inclusions

$$D_n(x) \cdot \frac{1}{x^k-1} \in \mathbb{Z}[x], \qquad k = 1, 2, \dots, n.$$

Mertens's formula ([17], [18], § 18.5, Theorem 330)

$$\sum_{l \leqslant n} \varphi(l) = \frac{3}{\pi^2} n^2 + O(n \log n),$$

which is the q-analogue of the prime number theorem in our case, leads to the following results.

Lemma 1 (see [10], § 2; [13], Lemma 2). *The following limit relation holds for each* $p \in \mathbb{Z} \setminus \{0, \pm 1\}$:

$$\lim_{n\to\infty} \frac{\log |D_n(p)|}{n^2} = \frac{3}{\pi^2} \log |p|.$$

Lemma 2 (see [11], Lemma 1). *The following limit relation holds for each integer* $p \in \mathbb{Z} \setminus \{0, \pm 1\}$ *and each half-open interval* $[u, v) \subset (0, 1)$ *with* $u, v \in \mathbb{Q}$:

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{l : \{n/l\} \in [u,v)} \log |\Phi_l(p)| = \frac{3}{\pi^2} \bigl(\psi'(u) - \psi'(v)\bigr) \log |p| = -\frac{3 \log |p|}{\pi^2} \int_u^v \mathrm{d}\psi'(z)$$

where $\{a\} = a - \lfloor a \rfloor$ *and* $\psi(z)$ *is the logarithmic derivative of the gamma function.*

§ 2. *q*-hypergeometric construction

We fix integer parameters

$$(\boldsymbol{a}, \boldsymbol{b}) = \begin{pmatrix} a_1, a_2, a_3 \\ b_1, b_2, b_3 \end{pmatrix} \tag{9}$$

satisfying the conditions

$$\min\{a_2, a_3\} \geqslant b_1 = 1, \quad a_2 < b_2, \quad a_3 < b_3, \quad a_1 + a_2 + a_3 \leqslant b_2 + b_3 - 1, \tag{10}$$

and consider the q-basic hypergeometric series [14]

$$\begin{aligned} G_q(\boldsymbol{a}, \boldsymbol{b}) &:= \frac{\Gamma_q(a_2)\,\Gamma_q(a_3)\,\Gamma_q(b_2-a_2)\,\Gamma_q(b_3-a_3)}{(1-q)^2 \Gamma_q(b_2)\,\Gamma_q(b_3)} \\ &\qquad \times {}_3\phi_2\left(\begin{matrix} q^{a_1}, q^{a_2}, q^{a_3} \\ q^{b_2}, q^{b_3} \end{matrix} \,\middle|\, q, q^{b_2+b_3-a_1-a_2-a_3} \right) \\ &= \frac{\Gamma_q(a_2)\,\Gamma_q(a_3)\,\Gamma_q(b_2-a_2)\,\Gamma_q(b_3-a_3)}{(1-q)^2 \Gamma_q(b_2)\,\Gamma_q(b_3)} \\ &\qquad \times \sum_{t=0}^{\infty} \frac{(q^{a_1}, q^{a_2}, q^{a_3}; q)_t}{(q^{b_1}, q^{b_2}, q^{b_3}; q)_t} q^{t(b_2+b_3-a_1-a_2-a_3)}, \end{aligned} \tag{11}$$

which is absolutely convergent in the region $|q| < 1$. The obvious symmetry of the quantity $G_q(\boldsymbol{a}, \boldsymbol{b})$ leads to the following result.

Lemma 3. *The quantity $G_q(\boldsymbol{a},\boldsymbol{b})$ is stable under the transformation*

$$\sigma\colon \begin{pmatrix} a_1, a_2, a_3 \\ 1, b_2, b_3 \end{pmatrix} \mapsto \begin{pmatrix} a_1, a_3, a_2 \\ 1, b_3, b_2 \end{pmatrix}.$$

Hall's identity ([14], formula (3.2.10))

$$\begin{aligned} &{}_3\phi_2\left(\begin{matrix} q^{a_1}, q^{a_2}, q^{a_3} \\ q^{b_2}, q^{b_3} \end{matrix} \,\middle|\, q, q^{b_2+b_3-a_1-a_2-a_3} \right) \\ &\quad= \frac{\Gamma_q(b_2)\,\Gamma_q(b_3)\,\Gamma_q(b_2+b_3-a_1-a_2-a_3)}{\Gamma_q(a_2)\,\Gamma_q(b_2+b_3-a_2-a_3)\,\Gamma_q(b_2+b_3-a_1-a_2)} \\ &\qquad\times {}_3\phi_2\left(\begin{matrix} q^{b_3-a_2}, & q^{b_2-a_2}, q^{b_2+b_3-a_1-a_2-a_3} \\ q^{b_2+b_3-a_2-a_3}, & q^{b_2+b_3-a_1-a_2} \end{matrix} \,\middle|\, q, q^{a_2} \right) \end{aligned}$$

gives rise to a 'non-trivial' transformation of the quantity $G_q(\boldsymbol{a},\boldsymbol{b})$.

Lemma 4. *The quantity $G_q(\boldsymbol{a},\boldsymbol{b})$ is stable under the transformation*

$$\tau\colon \begin{pmatrix} a_1, a_2, a_3 \\ 1, b_2, b_3 \end{pmatrix} \mapsto \begin{pmatrix} b_3-a_2, & b_2-a_2, b_2+b_3-a_1-a_2-a_3 \\ 1, b_2+b_3-a_2-a_3, & b_2+b_3-a_1-a_2 \end{pmatrix}.$$

The following statement presents recurrence relations for the quantity (11) that are q-extensions of the identities obtained in the proof of Theorem 2.1 in [6], p. 31.

Lemma 5. *The following identity holds*:

$$\begin{aligned} &q^{b_2+b_3-a_1-a_2-a_3} G_q\begin{pmatrix} a_1, a_2, a_3 \\ b_1, b_2, b_3 \end{pmatrix} \\ &\quad= G_q\begin{pmatrix} a_1, a_2-1, a_3-1 \\ b_1, b_2-1, b_3-1 \end{pmatrix} - G_q\begin{pmatrix} a_1-1, a_2-1, a_3-1 \\ b_1, b_2-1, b_3-1 \end{pmatrix}. \end{aligned} \tag{12}$$

Proof. Since

$$\begin{aligned} \frac{(q^{a_1};q)_{t+1}}{(q;q)_{t+1}} - \frac{(q^{a_1-1};q)_{t+1}}{(q;q)_{t+1}} &= \frac{(q^{a_1};q)_t}{(q;q)_t} \cdot \frac{(1-q^{a_1+\nu})-(1-q^{a_1-1})}{1-q^{\nu+1}} \\ &= q^{a_1-1}\frac{(q^{a_1};q)_t}{(q;q)_t}, \end{aligned}$$

it follows that

$$\begin{aligned} &\sum_{t=0}^{\infty} \frac{(q^{a_1}, q^{a_2-1}, q^{a_3-1};q)_t}{(q, q^{b_2-1}, q^{b_3-1};q)_t} q^{tc} - \sum_{t=0}^{\infty} \frac{(q^{a_1-1}, q^{a_2-1}, q^{a_3-1};q)_t}{(q, q^{b_2-1}, q^{b_3-1};q)_t} q^{tc} \\ &\quad= \sum_{t=0}^{\infty} \left(\frac{(q^{a_1}, q^{a_2-1}, q^{a_3-1};q)_{t+1}}{(q, q^{b_2-1}, q^{b_3-1};q)_{t+1}} - \frac{(q^{a_1-1}, q^{a_2-1}, q^{a_3-1};q)_{t+1}}{(q, q^{b_2-1}, q^{b_3-1};q)_{t+1}} \right) q^{(t+1)c} \\ &\quad= q^{c+a_1-1} \frac{(1-q^{a_2-1})(1-q^{a_3-1})}{(1-q^{b_2-1})(1-q^{b_3-1})} \sum_{t=0}^{\infty} \frac{(q^{a_1}, q^{a_2}, q^{a_3};q)_t}{(q, q^{b_2}, q^{b_3};q)_t} q^{tc}, \end{aligned} \tag{13}$$

where $c=b_2+b_3-a_1-a_2-a_3$. On the other hand, we have

$$\begin{aligned}
&\sum_{t=0}^{\infty}\frac{(q^{a_1-1},q^{a_2-1},q^{a_3-1};q)_t}{(q,q^{b_2-1},q^{b_3-1};q)_t}q^{tc}-\sum_{t=0}^{\infty}\frac{(q^{a_1-1},q^{a_2-1},q^{a_3-1};q)_t}{(q,q^{b_2-1},q^{b_3-1};q)_t}q^{t(c+1)}\\
&\quad=\sum_{t=0}^{\infty}\frac{(q^{a_1-1},q^{a_2-1},q^{a_3-1};q)_t}{(q,q^{b_2-1},q^{b_3-1};q)_t}(1-q^t)q^{tc}\\
&\quad=\sum_{t=0}^{\infty}\frac{(q^{a_1-1},q^{a_2-1},q^{a_3-1};q)_{t+1}}{(q,q^{b_2-1},q^{b_3-1};q)_{t+1}}(1-q^{t+1})q^{(t+1)c}\\
&\quad=q^c\frac{(1-q^{a_1-1})(1-q^{a_2-1})(1-q^{a_3-1})}{(1-q^{b_2-1})(1-q^{b_3-1})}\sum_{t=0}^{\infty}\frac{(q^{a_1},q^{a_2},q^{a_3};q)_t}{(q,q^{b_2},q^{b_3};q)_t}q^{tc}.
\end{aligned}\tag{14}$$

Summing the left-hand and the right-hand sides of relations (13) and (14) we obtain

$$\begin{aligned}
&\sum_{t=0}^{\infty}\frac{(q^{a_1},q^{a_2-1},q^{a_3-1};q)_t}{(q,q^{b_2-1},q^{b_3-1};q)_t}q^{tc}-\sum_{t=0}^{\infty}\frac{(q^{a_1-1},q^{a_2-1},q^{a_3-1};q)_t}{(q,q^{b_2-1},q^{b_3-1};q)_t}q^{t(c+1)}\\
&\quad=q^c\frac{(1-q^{a_2-1})(1-q^{a_3-1})}{(1-q^{b_2-1})(1-q^{b_3-1})}\sum_{t=0}^{\infty}\frac{(q^{a_1},q^{a_2},q^{a_3};q)_t}{(q,q^{b_2},q^{b_3};q)_t}q^{tc}.
\end{aligned}\tag{15}$$

Finally, multiplying both sides of equality (15) by

$$\frac{\Gamma_q(a_2-1)\,\Gamma_q(a_3-1)\,\Gamma_q(b_2-a_2)\,\Gamma_q(b_3-a_3)}{(1-q)^2\Gamma_q(b_2-1)\,\Gamma_q(b_3-1)}$$

we arrive at the required relation (12).

In the next section, we show that the quantities (11) so constructed are linear forms in 1 and $\zeta_q(2)$.

§ 3. Arithmetic of linear forms

We associate with the parameters (9) another collection $\boldsymbol{c}$ of ten integers:

$$\begin{aligned}
&c_{00}=(b_1+b_2+b_3)-(a_1+a_2+a_3)-2,\\
&c_{jk}=\begin{cases} a_j-b_k & \text{for } k=1,\\ b_k-a_j-1 & \text{for } k=2,3,\end{cases}\qquad j,k=1,2,3.
\end{aligned}\tag{16}$$

By (10) the set

$$\{c_{00},c_{21},c_{22},c_{33},c_{31}\}\tag{17}$$

consists of non-negative integers, while the integers in the set

$$\{c_{11},c_{23},c_{13},c_{12},c_{32}\}\tag{18}$$

can be negative. Note also that the old parameters (9) are uniquely recovered from the elements of either of the sets (17) and (18):

$$\begin{aligned} &a_1 = c_{22} + c_{33} - c_{00} + 1, \quad a_2 = c_{21} + 1, && a_3 = c_{31} + 1, \\ &b_1 = 1, \qquad\qquad b_2 = c_{21} + c_{22} + 2, && b_3 = c_{31} + c_{33} + 2; \end{aligned}$$

$$\begin{aligned} &a_1 = c_{11} + 1, \quad a_2 = c_{11} + c_{13} - c_{23} + 1, \quad a_3 = c_{11} + c_{12} - c_{32} + 1, \\ &b_1 = 1, \qquad b_2 = c_{11} + c_{12} + 2, \qquad\qquad b_3 = c_{11} + c_{13} + 2. \end{aligned}$$

By Lemmas 3, 4 and formulae (16) the action of the transformations σ and τ on the parameters $\boldsymbol{c}$ is as follows:

$$\sigma\colon \begin{pmatrix} c_{00}, c_{21}, c_{22}, c_{33}, c_{31} \\ c_{11}, c_{23}, c_{13}, c_{12}, c_{32} \end{pmatrix} \mapsto \begin{pmatrix} c_{00}, c_{31}, c_{33}, c_{22}, c_{21} \\ c_{11}, c_{32}, c_{12}, c_{13}, c_{23} \end{pmatrix}, \tag{19}$$

$$\tau\colon \begin{pmatrix} c_{00}, c_{21}, c_{22}, c_{33}, c_{31} \\ c_{11}, c_{23}, c_{13}, c_{12}, c_{32} \end{pmatrix} \mapsto \begin{pmatrix} c_{21}, c_{22}, c_{33}, c_{31}, c_{00} \\ c_{23}, c_{13}, c_{12}, c_{32}, c_{11} \end{pmatrix}. \tag{20}$$

Thus the transformations σ and τ rearrange the 10-element set

$$\boldsymbol{c} := \begin{pmatrix} c_{00}, c_{21}, c_{22}, c_{33}, c_{31} \\ c_{11}, c_{23}, c_{13}, c_{12}, c_{32} \end{pmatrix}, \tag{21}$$

but do not change the quantity

$$H_q(\boldsymbol{c}) := G_q(\boldsymbol{a}, \boldsymbol{b}). \tag{22}$$

Denote by $\mathfrak{G}_0 \subset \mathfrak{S}_{10}$ the group generated by the permutations (19) and (20); since the order of σ is 5 and the order of τ is 2, the group $\mathfrak{G}_0$ contains ten elements.

Note also that formulae (16) allow one to write the recurrence relations of Lemma 5 in the following form:

$$\begin{aligned} q^{c_{00}+1} H_q(\boldsymbol{c}) = H_q &\begin{pmatrix} c_{00}, c_{21} - 1, & c_{22}, & c_{33}, & c_{31} - 1 \\ c_{11}, \quad c_{23}, & c_{13} - 1, & c_{12} - 1, & c_{32} \end{pmatrix} \\ &- H_q \begin{pmatrix} c_{00} + 1, c_{21} - 1, c_{22}, c_{33}, c_{31} - 1 \\ c_{11} - 1, \quad c_{23}, \quad c_{13}, c_{12}, \quad c_{32} \end{pmatrix}. \end{aligned} \tag{23}$$

To each set of parameters (9) and the corresponding set (16) we assign the quantity

$$\begin{aligned} m = m(\boldsymbol{c}) &:= c_{00} + c_{21} + c_{22} + c_{33} + c_{31} = c_{11} + c_{23} + c_{13} + c_{12} + c_{32} \\ &= 2(b_1 + b_2 + b_3) - (a_1 + a_2 + a_3) - 3; \end{aligned} \tag{24}$$

we shall denote by $m_1 = m_1(\boldsymbol{c})$ and $m_2 = m_2(\boldsymbol{c})$ the two successive maxima (located at distinct places in the set (18); see also [7], p. 273); the fact that $m_1 \geqslant 0$ and $m_2 \geqslant 0$ is proved in [6], Theorem 2.1. We set

$$\begin{gathered} M_0 = M_0(\boldsymbol{c}) := \begin{cases} c_{00}c_{21} + c_{31}c_{33} - c_{21}c_{33} & \text{if } c_{21} \leqslant c_{31}, \\ c_{00}c_{31} + c_{21}c_{22} - c_{31}c_{22} & \text{if } c_{21} \geqslant c_{31}, \end{cases} = M_0(\sigma\boldsymbol{c}), \\ M = M(\boldsymbol{c}) := \max_{\mathfrak{g}\in\mathfrak{G}_0}\{M_0(\mathfrak{g}\boldsymbol{c})\} = \max_{0\leqslant j\leqslant 4}\{M_0(\tau^j\boldsymbol{c})\} \geqslant 0, \end{gathered} \tag{25}$$

where $\mathfrak{g}\boldsymbol{c}$ denotes the action of the permutation $\mathfrak{g} \in \mathfrak{G}_0$ on the set (21). The above definitions ensure the stability of the quantities

$$H_q(\boldsymbol{c}), \quad m(\boldsymbol{c}), \quad m_1(\boldsymbol{c}), \quad m_2(\boldsymbol{c}), \quad M(\boldsymbol{c})$$

under the action of the group $\mathfrak{G}_0$.

Note that $\zeta_q(2)$ is not a rational (nor even algebraic) function over $\mathbb{C}(q)$.

Proposition 1. *The following inclusion holds*:

$$x^{-M}\cdot D_{m_1}(x)D_{m_2}(x)\cdot H_q(\boldsymbol{c})\in\mathbb{Z}[x]\zeta_q(2)+\mathbb{Z}[x], \qquad \textit{where} \quad x=q^{-1}. \tag{26}$$

Proof. We prove the lemma by induction on the quantity

$$m(\boldsymbol{c})=c_{00}+c_{21}+c_{22}+c_{33}+c_{31}; \tag{27}$$

note that each term in (27) is non-negative.

As the induction basis we take the cases when at least three of the parameters (17) are zero. Two cases are possible: the three vanishing parameters either succeed or do not succeed one another in the cyclic set (17) (that is, we mean that the parameter c_{00} succeeds c_{31}). With the help of several applications of the cyclic permutation (19) the first case can be reduced to

$$c_{22}=c_{33}=c_{31}=0, \tag{28}$$

and the second case to

$$c_{00}=c_{22}=c_{33}=0; \tag{29}$$

a direct computation shows that $M(\boldsymbol{c})=0$ in both cases. We start with the second case.

If (29) holds, then

$$c_{11}=c_{22}+c_{33}-c_{00}=0;$$

hence

$$a_1=c_{11}+1=1, \quad b_2=c_{22}+a_2+1=a_2+1, \quad b_3=c_{33}+a_3+1=a_3+1.$$

Consequently,

$$\begin{aligned} H_q(\boldsymbol{c})&=G_q\begin{pmatrix}1, & a_2, & a_3\\ 1, & a_2+1, & a_3+1\end{pmatrix}\\ &=\frac{\Gamma_q(a_2)\,\Gamma_q(a_3)}{(1-q)^2\Gamma_q(a_2+1)\,\Gamma_q(a_3+1)}\cdot {}_3\phi_2\left(\begin{matrix}q, & q^{a_2}, & q^{a_3}\\ & q^{a_2+1}, & q^{a_3+1}\end{matrix}\,\middle|\, q,q\right)\\ &=\frac{1}{(1-q^{a_2})(1-q^{a_3})}\sum_{t=0}^{\infty}\frac{(q^{a_2},q^{a_3};q)_t}{(q^{a_2+1},q^{a_3+1};q)_t}q^t\\ &=\sum_{t=0}^{\infty}\frac{q^t}{(1-q^{t+a_2})(1-q^{t+a_3})}. \end{aligned}$$

If $a_2=a_3$, then

$$\begin{aligned} H_q(\boldsymbol{c})&=q^{-a_2}\sum_{t=0}^{\infty}\frac{q^{t+a_2}}{(1-q^{t+a_2})^2}=q^{-a_2}\biggl(\sum_{n=1}^{\infty}-\sum_{n=1}^{a_2-1}\biggr)\frac{q^n}{(1-q^n)^2}\\ &=q^{-a_2}\zeta_q(2)-q^{-a_2}\sum_{n=1}^{a_2-1}\frac{q^n}{(1-q^n)^2}=x^{a_2}\zeta_q(2)-x^{a_2}\sum_{n=1}^{a_2-1}\frac{x^n}{(x^n-1)^2}, \end{aligned}$$

which yields the inclusion

$$x^{-a_2} \cdot D_{a_2-1}(x)^2 \cdot H_q(\boldsymbol{c}) \in \mathbb{Z}[x]\zeta_q(2) + \mathbb{Z}[x]. \tag{30}$$

If $a_2 \neq a_3$, then (assuming without loss of generality that $a_2 < a_3$)

$$\begin{aligned} H_q(\boldsymbol{c}) &= \frac{1}{q^{a_2} - q^{a_3}} \sum_{t=0}^{\infty} \left(\frac{1}{1-q^{t+a_2}} - \frac{1}{1-q^{t+a_3}} \right) = \frac{1}{q^{a_2} - q^{a_3}} \sum_{n=a_2}^{a_3-1} \frac{1}{1-q^n} \\ &= \frac{x^{a_3}}{x^{a_3-a_2} - 1} \cdot x^{a_2} \sum_{n=a_2}^{a_3-1} \frac{x^{n-a_2}}{x^n - 1}, \end{aligned}$$

therefore

$$x^{-(a_2+a_3)} \cdot D_{a_3-a_2}(x) D_{a_3-1}(x) \cdot H_q(\boldsymbol{c}) \in \mathbb{Z}[x]. \tag{31}$$

The inclusions (30) and (31) mean that the required relation (26) holds in the case (29) because

$$\{c_{11}, c_{23}, c_{13}, c_{12}, c_{32}\} = \{0, a_3 - a_2, a_3 - 1, a_2 - 1, a_2 - a_3\}.$$

Consider now the case (28). The situation when $c_{00} = 0$ was investigated above, therefore we assume that $c_{00} > 0$, so that $a_1 \leqslant 0$ and the series in (11) terminates. In the same vein, without loss of generality we assume that $c_{21} > 0$ (otherwise, applying the permutation τ we return to the case (29)) and, as a consequence, $a_2 > 1$. We have

$$b_2 = c_{22} + a_2 + 1 = a_2 + 1, \quad a_3 = c_{31} + 1 = 1, \quad b_3 = c_{33} + a_3 + 1 = 2,$$

which yields

$$\begin{aligned} H_q(\boldsymbol{c}) &= G_q \begin{pmatrix} a_1, & a_2, & 1 \\ 1, & a_2+1, & 2 \end{pmatrix} \\ &= \frac{\Gamma_q(a_2)}{(1-q)^2 \Gamma_q(a_2+1)} \cdot {}_3\phi_2 \left(\begin{matrix} q^{a_1}, & q^{a_2}, & q \\ & q^{a_2+1}, & q^2 \end{matrix} \,\middle|\, q, q^{-a_1+2} \right) \\ &= \frac{1}{(1-q)(1-q^{a_2})} \sum_{t=0}^{\infty} \frac{(q^{a_1}, q^{a_2}; q)_t}{(q^2, q^{a_2+1}; q)_t} q^{t(-a_1+2)}. \end{aligned} \tag{32}$$

Setting $n = -a_1 \geqslant 0$ we write the last series as follows:

$$\begin{aligned} H_q(\boldsymbol{c}) &= \sum_{t=0}^{n-1} \frac{(q^{-n}; q)_t}{(q; q)_{t+1}} \cdot \frac{q^{t(n+2)}}{1 - q^{t+a_2}} = \sum_{t=0}^{n-1} \frac{(q^{-n}; q)_t}{(q; q)_t} \cdot \frac{q^{t(n+2)}}{(1-q^{t+1})(1-q^{t+a_2})} \\ &= \sum_{t=0}^{n-1} (-1)^t x^{t(t+1)/2} \begin{bmatrix} n \\ t \end{bmatrix}_x \cdot \frac{x^{(t+1)+(t+a_2)-t(n+2)}}{(x^{t+1}-1)(x^{t+a_2}-1)} \\ &= x^{a_2+1-n(n+1)/2} \sum_{t=0}^{n-1} (-1)^t \begin{bmatrix} n \\ t \end{bmatrix}_x \cdot \frac{x^{(n-t)(n-t-1)/2}}{(x^{t+1}-1)(x^{t+a_2}-1)}. \end{aligned} \tag{33}$$

By (8) formula (33) leads to the inclusion

$$x^{-(a_2+1)+n(n+1)/2} \cdot D_n(x) D_{a_2+n-1}(x) \cdot H_q(\boldsymbol{c}) \in \mathbb{Z}[x]. \tag{34}$$

On the other hand, the sum on the right-hand side of (32) has another representation:

$$\begin{aligned} H_q(\boldsymbol{c}) &= \frac{q^{a_1-2}}{(1-q^{a_1-1})(1-q^{a_2-1})} \sum_{t=0}^{\infty} \frac{(q^{a_1-1}, q^{a_2-1}; q)_{t+1}}{(q, q^{a_2}; q)_{t+1}} q^{(t+1)(-a_1+2)} \\ &= \frac{q^{a_1-2}}{(1-q^{a_1-1})(1-q^{a_2-1})} \left({}_2\phi_1 \left(\begin{matrix} q^{a_1-1}, q^{a_2-1} \\ q^{a_2} \end{matrix} \,\middle|\, q, q^{-a_1+2} \right) - 1 \right). \end{aligned}$$

Next, Heine's q-analogue ([14], formula (1.5.2)) of Gauss's summation formula allows one to represent the last q-basic series in closed form:

$$\begin{aligned} H_q(\boldsymbol{c}) &= \frac{q^{a_1-2}}{(1-q^{a_1-1})(1-q^{a_2-1})} \left(\frac{(q;q)_{-a_1+1}}{(q^{a_2};q)_{-a_1+1}} - 1 \right) \\ &= -q^{-1} \frac{(q;q)_n}{(q^{a_2-1};q)_{n+2}} - \frac{q^{-n-2}}{(1-q^{-n-1})(1-q^{a_2-1})} \\ &= -x^{a_2(n+2)+n} \frac{(x;x)_n}{(x^{a_2-1};x)_{n+2}} - \frac{x^{a_2+n+1}}{(1-x^{n+1})(x^{a_2-1}-1)}, \end{aligned}$$

which proves the inclusion

$$x^{-(a_2+n+1)} \cdot (x^{n+1}-1)(x;x)_{a_2+n+1} \cdot H_q(\boldsymbol{c}) \in \mathbb{Z}[x]. \tag{35}$$

Since the polynomial x is coprime with the polynomials $D_n(x), D_{a_2+n-1}(x), x^{n+1}-1$, and $(x;x)_{a_2+n+1}$, the inclusions (34), (35) can be written as follows:

$$x^{-(a_2+n+1)} \cdot D_n(x) D_{a_2+n-1}(x) \cdot H_q(\boldsymbol{c}) \in \mathbb{Z}[x]$$

or, equivalently, returning to the parameter $a_1 = -n$,

$$x^{-(a_2-a_1+1)} \cdot D_{-a_1}(x) D_{a_2-a_1-1}(x) \cdot H_q(\boldsymbol{c}) \in \mathbb{Z}[x].$$

From the relation

$$\{c_{11}, c_{23}, c_{13}, c_{12}, c_{32}\} = \{a_1-1, -a_2+1, -a_1+1, a_2-a_1, a_2-1\},$$

we see that the required inclusion (26) holds again. We have thus established the induction basis.

Assume that at least three elements of the set (17) are non-zero and the required inclusion (26) has been proved for each set $\boldsymbol{c}'$ with $m(\boldsymbol{c}') < m(\boldsymbol{c})$.

We consider the case $M(\boldsymbol{c}) = 0$ first. Among the three non-zero parameters we can always take two that do not succeed one another in the cyclic set (17); the cyclic permutation (20) allows one to work with an $\mathfrak{G}_0$-equivalent set $\boldsymbol{c}$ in which $c_{21} > 0$

and $c_{31}>0$. Then the inclusion (26) follows from the recurrence relations (23) and the induction hypothesis, because for the sets

$$\begin{aligned} \boldsymbol{c}' &= \begin{pmatrix} c_{00}, c_{21}-1, & c_{22}, & c_{33}, & c_{31}-1 \\ c_{11}, & c_{23}, & c_{13}-1, c_{12}-1, & c_{32} \end{pmatrix}, \\ \boldsymbol{c}'' &= \begin{pmatrix} c_{00}+1, c_{21}-1, c_{22}, c_{33}, c_{31}-1 \\ c_{11}-1, & c_{23}, & c_{13}, c_{12}, & c_{32} \end{pmatrix} \end{aligned} \tag{36}$$

we have

$$m_1(\boldsymbol{c}') \leqslant m_1(\boldsymbol{c}), \quad m_2(\boldsymbol{c}') \leqslant m_2(\boldsymbol{c}), \qquad m_1(\boldsymbol{c}'') \leqslant m_1(\boldsymbol{c}), \quad m_2(\boldsymbol{c}'') \leqslant m_2(\boldsymbol{c}), \tag{37}$$

and $M(\boldsymbol{c}') \geqslant 0$, $M(\boldsymbol{c}'') \geqslant 0$.

Now let $M(\boldsymbol{c})>0$. In view of the $\mathfrak{G}_0$-stability of the quantity $M(\boldsymbol{c})$ we may assume that $M_0(\boldsymbol{c}) = M(\boldsymbol{c})$. If, in addition, $c_{21}>0$ and $c_{31}>0$, then applying identities (23) as before in combination with the induction hypothesis for the sets (36) and bearing in mind relations (37) and

$$\begin{aligned} M(\boldsymbol{c}') &\geqslant M_0(\boldsymbol{c}') = M_0(\boldsymbol{c}) - c_{00} = M(\boldsymbol{c}) - c_{00}, \\ M(\boldsymbol{c}'') &\geqslant M_0(\boldsymbol{c}'') = M_0(\boldsymbol{c}) - c_{00} + \min\{c_{21}-1, c_{31}-1\} \\ &\geqslant M_0(\boldsymbol{c}) - c_{00} = M(\boldsymbol{c}) - c_{00} \end{aligned}$$

we arrive at the required inclusion (26) again.

Assume that in the set $\boldsymbol{c}$ with $M_0(\boldsymbol{c}) = M(\boldsymbol{c}) > 0$ at least one of the parameters c_{21} and c_{31} is non-zero. Both parameters cannot vanish simultaneously because if $c_{21}=c_{31}=0$, then $M_0(\boldsymbol{c})=0$. Without loss of generality we may assume that $c_{21}=0$ and $c_{31}>0$, since $M_0(\boldsymbol{c}) = M_0(\sigma\boldsymbol{c})$. As follows from the definition (25), in our case $M(\boldsymbol{c}) = M_0(\boldsymbol{c}) = c_{31}c_{33} > 0$, which, in particular, yields $c_{33}>0$. At least one of the parameters c_{00} and c_{22} is non-zero because $c_{21}=0$ and the set (17) contains at most two parameters equal to zero. If $c_{00}>0$, then we consider identity (23) for the set $\tau^4\boldsymbol{c}$:

$$x^{-c_{31}-1} H_q(\tau^4\boldsymbol{c}) = H_q(\boldsymbol{c}') - H_q(\boldsymbol{c}''), \tag{38}$$

where

$$\begin{aligned} \boldsymbol{c}' &= \begin{pmatrix} c_{31}, c_{00}-1, & c_{21}, & c_{22}, & c_{33}-1 \\ c_{32}, & c_{11}, & c_{23}-1, c_{13}-1, & c_{12} \end{pmatrix}, \\ \boldsymbol{c}'' &= \begin{pmatrix} c_{31}+1, c_{00}-1, c_{21}, c_{22}, c_{33}-1 \\ c_{32}-1, & c_{11}, & c_{23}, c_{13}, & c_{12} \end{pmatrix}. \end{aligned} \tag{39}$$

From the $\mathfrak{G}_0$-stability of the quantities $m_1(\boldsymbol{c})$ and $m_2(\boldsymbol{c})$ for the sets (39) we deduce the estimates (37). Moreover,

$$\begin{aligned} M(\boldsymbol{c}') &\geqslant M_0(\tau\boldsymbol{c}') = c_{31}(c_{33}-1) = M(\boldsymbol{c}) - c_{31}, \\ M(\boldsymbol{c}'') &\geqslant M_0(\tau\boldsymbol{c}'') = (c_{31}+1)(c_{33}-1) \geqslant M(\boldsymbol{c}) - c_{31}, \end{aligned}$$

which in combination with the induction hypothesis for the sets (39) and identity (38) yields the inclusion (26). If $c_{22}>0$, then similar arguments with $\tau^4\boldsymbol{c}$ replaced by $\tau^3\boldsymbol{c}$ also yield the inclusion (26). This completes the proof of the induction step and Proposition 1.

§ 4. Group structure for $\zeta_q(2)$

We shall require stronger restrictions on the parameters (9) than in (10):

$$\{b_1 = 1\} \leqslant \{a_1, a_2, a_3\} < \{b_2, b_3\}, \quad a_1 + a_2 + a_3 \leqslant b_1 + b_2 + b_3 - 2. \tag{40}$$

Then all the parameters (16) are non-negative and in addition to the transformations σ and τ one can consider, on the one hand, all possible permutations of the parameters a_1, a_2, a_3 and, on the other hand, the transposition of the parameters b_2 and b_3. These permutations do not change the quantity

$$\frac{\Gamma_q(a_1)}{\Gamma_q(b_2 - a_2)\,\Gamma_q(b_3 - a_3)} \cdot G_q(\boldsymbol{a}, \boldsymbol{b}) = \frac{[c_{11}]_q!}{[c_{22}]_q!\,[c_{33}]_q!} \cdot H_q(\boldsymbol{c}). \tag{41}$$

Hence (41) is stable under the action of the '$(\boldsymbol{a}, \boldsymbol{b})$-trivial' group $\mathfrak{G}_1$ generated by the permutations

$$\mathfrak{a}_1 = (a_1\ a_3), \quad \mathfrak{a}_2 = (a_1\ a_3), \quad \mathfrak{b} = (b_2\ b_3). \tag{42}$$

In terms of the parameters $\boldsymbol{c}$ the actions of the permutations (42) can be expressed as follows:

$$\begin{gathered} \mathfrak{a}_1 = (c_{11}\ c_{31})(c_{12}\ c_{32})(c_{13}\ c_{33}), \quad \mathfrak{a}_2 = (c_{21}\ c_{31})(c_{22}\ c_{32})(c_{23}\ c_{33}), \\ \mathfrak{b} = (c_{12}\ c_{13})(c_{22}\ c_{23})(c_{32}\ c_{33}). \end{gathered} \tag{43}$$

Regarding the group

$$\mathfrak{G} = \langle \mathfrak{G}_0, \mathfrak{G}_1 \rangle = \langle \sigma, \tau, \mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{b} \rangle$$

as a permutation group of the 10-element set $\boldsymbol{c}$ and bearing in mind the $\mathfrak{G}_0$-stability of the quantity (22) and the $\mathfrak{G}_1$-stability of the quantity (41) we arrive at the following result.

Lemma 6 (cf. [6], § 3; [9], Lemma 14). *The quantity*

$$\frac{H_q(\boldsymbol{c})}{\Pi_q(\boldsymbol{c})}, \qquad \textit{where } \Pi_q(\boldsymbol{c}) = [c_{00}]_q!\,[c_{21}]_q!\,[c_{22}]_q!\,[c_{33}]_q!\,[c_{31}]_q!,$$

is stable under the action of the group $\mathfrak{G}$.

Note that the quantity (24) is also $\mathfrak{G}$-stable.

It is proved in [6] that the group $\mathfrak{G} \subset \mathfrak{S}_{10}$ has order 120. The second-order permutations (43) and

$$\mathfrak{h} = (c_{00}\ c_{22})(c_{11}\ c_{33})(c_{13}\ c_{31})$$

can be taken for generators of this group ([9], § 6); then we have $\sigma = \mathfrak{a}_2\,\mathfrak{b}$ and $\tau = (\mathfrak{a}_2\,\mathfrak{a}_1\,\mathfrak{b}\,\mathfrak{h})^2$.

With the help of the easily verified identity

$$[n]_q! = x^{-n(n-1)/2}[n]_x!, \qquad \text{where } x = q^{-1},$$

Lemma 6 yields the $\mathfrak{G}$-stability of the quantity

$$\frac{H_q(\boldsymbol{c})}{x^{-N(\boldsymbol{c})}\Pi_x(\boldsymbol{c})}, \tag{44}$$

where we set

$$N(\boldsymbol{c}) := \frac{c_{00}(c_{00}-1) + c_{21}(c_{21}-1) + c_{22}(c_{22}-1) + c_{33}(c_{33}-1) + c_{31}(c_{31}-1)}{2}. \tag{45}$$

Lemma 7. *The quantity $M(\boldsymbol{c})+N(\boldsymbol{c})$ is stable under the action of the group $\mathfrak{G}$.*

Proof. Of course, this can be proved using the definitions (25) and (45) of the quantities $M(\boldsymbol{c})$ and $N(\boldsymbol{c})$, respectively. The cumbersome brute force examination arising in this way makes the proof rather boring, therefore we content ourselves with mentioning that it is enough to verify the statement for a 'sufficiently general' set of parameters $(\boldsymbol{a},\boldsymbol{b})$ and the corresponding set $\boldsymbol{c}$. Using a special program for the `PARI-GP` calculator we have verified the claim of the lemma for all sets of parameters (9) satisfying conditions (40) such that $b_2+b_3\leqslant 100$. This completes the proof of the lemma.

Denoting by m_1^* and m_2^*, $m_1^*\geqslant m_2^*$, the two successive maxima of the 10-element set $\boldsymbol{c}$, by Proposition 1 we obtain the inclusion

$$x^{-M}\cdot D_{m_1^*}(x)D_{m_2^*}(x)\cdot H_q(\boldsymbol{c})\in\mathbb{Z}[x]\zeta_q(2)+\mathbb{Z}[x],\qquad \text{where } x=q^{-1}. \tag{46}$$

Moreover, the quantities m_1^* and m_2^* (by contrast with the quantities m_1 and m_2 introduced in § 3) are $\mathfrak{G}$-stable.

For a fixed set of parameters $(\boldsymbol{a},\boldsymbol{b})$ satisfying conditions (40) and for the corresponding set (16) we consider now the polynomial

$$\Omega(x):=\prod_{l=1}^{m_1^*}\Phi_l^{\nu_l}(x)\in\mathbb{Z}[x],$$

where

$$\nu_l:=\max_{\mathfrak{g}\in\mathfrak{G}}\operatorname{ord}_{\Phi_l(x)}\frac{\Pi_x(\boldsymbol{c})}{\Pi_x(\mathfrak{g}\boldsymbol{c})},\qquad l=1,2,\dots\,. \tag{47}$$

Proposition 2. *The following inclusion holds*:

$$x^{-M}\cdot D_{m_1^*}(x)D_{m_2^*}(x)\cdot\Omega^{-1}(x)\cdot H_q(\boldsymbol{c})\in\mathbb{Z}[x]\zeta_q(2)+\mathbb{Z}[x],\qquad \textit{where } x=q^{-1}. \tag{48}$$

Proof. In view of the $\mathfrak{G}$-stability of the quantities $M(\boldsymbol{c})+N(\boldsymbol{c})$, $m_1^*(\boldsymbol{c})$, $m_2^*(\boldsymbol{c})$, and (44), we conclude from the inclusion (46) that for each permutation $\mathfrak{g}\in\mathfrak{G}$ the linear form

$$\begin{aligned}
&x^{-M(\boldsymbol{c})}\cdot D_{m_1^*(\boldsymbol{c})}(x)D_{m_2^*(\boldsymbol{c})}(x)\cdot\frac{\Pi_x(\mathfrak{g}\boldsymbol{c})}{\Pi_x(\boldsymbol{c})}\cdot H_q(\boldsymbol{c})\\
&\quad=x^{-M(\boldsymbol{c})-N(\boldsymbol{c})+N(\mathfrak{g}\boldsymbol{c})}\cdot D_{m_1^*(\boldsymbol{c})}(x)D_{m_2^*(\boldsymbol{c})}(x)\cdot H_q(\mathfrak{g}\boldsymbol{c})\\
&\quad=x^{-M(\mathfrak{g}\boldsymbol{c})}\cdot D_{m_1^*(\mathfrak{g}\boldsymbol{c})}(x)D_{m_2^*(\mathfrak{g}\boldsymbol{c})}(x)\cdot H_q(\mathfrak{g}\boldsymbol{c})
\end{aligned}$$

belongs to $\mathbb{Z}[x]\zeta_q(2)+\mathbb{Z}[x]$. We now recall that $\zeta_q(2)$ as a function of $x=q^{-1}$ is irrational over $\mathbb{Q}(x)$, and that the factorization of $\Pi_x(\mathfrak{g}\boldsymbol{c})$, $\mathfrak{g}\in\mathfrak{G}$, in a product of irreducible polynomials contains only polynomials (4). Using the inequalities

$$\nu_l=0\quad\text{for } l>m_1^*,\qquad \nu_l\leqslant 1\quad\text{for } m_2^*<l\leqslant m_1^*$$

proved in [6] we obtain the required inclusion (48). The proof of the proposition is complete.

By the $\mathfrak{G}_0$-stability of the quantity $H_q(\boldsymbol{c})$ it is sufficient to consider the action of representatives of left cosets of the quotient $\mathfrak{G}/\mathfrak{G}_0$ (of order 12) in the (ordered) 5-element set $\boldsymbol{c}' = (c_{00}, c_{21}, c_{22}, c_{33}, c_{31})$ to determine the exponents (47): we have

$$\begin{aligned}
\nu_l &= \max_{\mathfrak{g}\in\mathfrak{G}/\mathfrak{G}_0} \operatorname{ord}_{\Phi_l(x)} \frac{\Pi_x(\boldsymbol{c})}{\Pi_x(\mathfrak{g}\boldsymbol{c})} \\
&= \max_{\mathfrak{g}\in\mathfrak{G}/\mathfrak{G}_0} \biggl(\biggl\lfloor \frac{c_{00}}{l} \biggr\rfloor + \biggl\lfloor \frac{c_{21}}{l} \biggr\rfloor + \biggl\lfloor \frac{c_{22}}{l} \biggr\rfloor + \biggl\lfloor \frac{c_{33}}{l} \biggr\rfloor + \biggl\lfloor \frac{c_{31}}{l} \biggr\rfloor \\
&\qquad - \biggl\lfloor \frac{\mathfrak{g}c_{00}}{l} \biggr\rfloor - \biggl\lfloor \frac{\mathfrak{g}c_{21}}{l} \biggr\rfloor - \biggl\lfloor \frac{\mathfrak{g}c_{22}}{l} \biggr\rfloor - \biggl\lfloor \frac{\mathfrak{g}c_{33}}{l} \biggr\rfloor - \biggl\lfloor \frac{\mathfrak{g}c_{31}}{l} \biggr\rfloor \biggr), \qquad l = 1, 2, \dots, m_1^*,
\end{aligned} \tag{49}$$

by (6). We take the following representatives:

$$\begin{array}{llll}
\mathfrak{g}_0 = \mathrm{id}, & \mathfrak{g}_1 = \mathfrak{a}_1\,\mathfrak{a}_2\,\mathfrak{a}_1, & \mathfrak{g}_2 = \mathfrak{a}_1, & \mathfrak{g}_3 = \mathfrak{a}_2, \\
\mathfrak{g}_4 = \mathfrak{a}_1\,\mathfrak{a}_2, & \mathfrak{g}_5 = \mathfrak{a}_2\,\mathfrak{a}_1, & \mathfrak{g}_6 = \mathfrak{h}\,\mathfrak{a}_1\,\mathfrak{a}_2\,\mathfrak{a}_1, & \mathfrak{g}_7 = \mathfrak{h}\,\mathfrak{a}_2, \\
\mathfrak{g}_8 = \mathfrak{h}\,\mathfrak{a}_1\,\mathfrak{a}_2, & \mathfrak{g}_9 = \mathfrak{h}\,\mathfrak{a}_2\,\mathfrak{a}_1, & \mathfrak{g}_{10} = \mathfrak{h}\,\mathfrak{a}_1\,\mathfrak{a}_2\,\mathfrak{a}_1\,\mathfrak{h}\,\mathfrak{a}_2, & \mathfrak{g}_{11} = \mathfrak{h}\,\mathfrak{a}_1\,\mathfrak{a}_2\,\mathfrak{a}_1\,\mathfrak{h}\,\mathfrak{a}_2\,\mathfrak{a}_1.
\end{array} \tag{50}$$

Then

$$\begin{aligned}
\mathfrak{g}_0\boldsymbol{c}' &= (c_{00}, c_{21}, c_{22}, c_{33}, c_{31}), & \mathfrak{g}_1\boldsymbol{c}' &= (c_{00}, c_{11}, c_{12}, c_{33}, c_{31}), \\
\mathfrak{g}_2\boldsymbol{c}' &= (c_{00}, c_{21}, c_{22}, c_{13}, c_{11}), & \mathfrak{g}_3\boldsymbol{c}' &= (c_{00}, c_{31}, c_{32}, c_{23}, c_{21}), \\
\mathfrak{g}_4\boldsymbol{c}' &= (c_{00}, c_{11}, c_{12}, c_{23}, c_{21}), & \mathfrak{g}_5\boldsymbol{c}' &= (c_{00}, c_{31}, c_{32}, c_{13}, c_{11}), \\
\mathfrak{g}_6\boldsymbol{c}' &= (c_{22}, c_{33}, c_{12}, c_{11}, c_{13}), & \mathfrak{g}_7\boldsymbol{c}' &= (c_{22}, c_{13}, c_{32}, c_{23}, c_{21}), \\
\mathfrak{g}_8\boldsymbol{c}' &= (c_{22}, c_{33}, c_{12}, c_{23}, c_{21}), & \mathfrak{g}_9\boldsymbol{c}' &= (c_{22}, c_{13}, c_{32}, c_{31}, c_{33}), \\
\mathfrak{g}_{10}\boldsymbol{c}' &= (c_{12}, c_{23}, c_{32}, c_{31}, c_{33}), & \mathfrak{g}_{11}\boldsymbol{c}' &= (c_{12}, c_{23}, c_{32}, c_{13}, c_{11}).
\end{aligned} \tag{51}$$

§ 5. Evaluation of linear forms and their coefficients

In this section we assume that the set of integer parameters (9) satisfies conditions (40). Using the following explicit expression for the quantity (22):

$$\begin{gathered}
G_q(\boldsymbol{a}, \boldsymbol{b}) = A\zeta_q(2) - B, \\
A = A_q(\boldsymbol{a}, \boldsymbol{b}) = A_q(\boldsymbol{c}) \in \mathbb{Q}(q), \quad B = B_q(\boldsymbol{a}, \boldsymbol{b}) = B_q(\boldsymbol{c}) \in \mathbb{Q}(q),
\end{gathered} \tag{52}$$

we shall evaluate $|G_q(\boldsymbol{a}, \boldsymbol{b})|$ and $|A|$ for $|q| \leqslant 1/2$.

We start from a new representation for the quantity (11). Namely, consider the function

$$\begin{aligned}
R_q(t) = R_q(\boldsymbol{a}, \boldsymbol{b}; t) &:= \frac{\Gamma_q(b_2 - a_2)\,\Gamma_q(b_3 - a_3)}{(1-q)^2\Gamma_q(a_1 - b_1 + 1)} \cdot q^{t(b_1+b_2+b_3-a_1-a_2-a_3-2)} \\
&\quad \times \frac{\Gamma_q(t + a_1)\,\Gamma_q(t + a_2)\,\Gamma_q(t + a_3)}{\Gamma_q(t + b_1)\,\Gamma_q(t + b_2)\,\Gamma_q(t + b_3)}
\end{aligned} \tag{53}$$

and write formula (11) as follows:

$$G_q(\boldsymbol{a}, \boldsymbol{b}) = \sum_{t=0}^{\infty} R_q(t) q^t. \tag{54}$$

Proposition 3. *Suppose that* $c_{00} = b_2 + b_3 - a_1 - a_2 - a_3 - 1 \geqslant 5$ *and* $|q| \leqslant 1/2$. *Then the estimates*

$$3^{-3(b_2+b_3)} < |G_q(\boldsymbol{a},\boldsymbol{b})| < 3^{3(b_2+b_3)} \tag{55}$$

hold.

Proof. From the functional equation

$$\Gamma_q(t+1) = \frac{1-q^t}{1-q}\Gamma_q(t) \tag{56}$$

for the Γ_q-function we deduce

$$\frac{R_q(t+1)}{R_q(t)} = \frac{(1-q^{t+a_1})(1-q^{t+a_2})(1-q^{t+a_3})}{(1-q^{t+b_1})(1-q^{t+b_2})(1-q^{t+b_3})} \cdot q^{c_{00}};$$

hence

$$\frac{|R_q(t+1)q^{t+1}|}{|R_q(t)q^t|} \leqslant \frac{(1+|q|)^3}{(1-|q|)^3} \cdot |q|^{c_{00}+1} \leqslant 3^3 \cdot 2^{-(c_{00}+1)} < \frac{1}{2}. \tag{57}$$

Applying the estimate (57) to the series in (54) we obtain

$$\begin{aligned}|G_q(\boldsymbol{a},\boldsymbol{b})| &\leqslant |R_q(0)| \cdot \left(1 + \frac{|R_q(1)q|}{|R_q(0)|} + \frac{|R_q(2)q^2|}{|R_q(0)|} + \frac{|R_q(3)q^3|}{|R_q(0)|} + \cdots\right) \\ &< |R_q(0)| \cdot \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right) = 2|R_q(0)| \end{aligned} \tag{58}$$

and, on the other hand,

$$|G_q(\boldsymbol{a},\boldsymbol{b})| \geqslant |R_q(0)| \cdot \left(1 - \frac{|R_q(1)q|}{|R_q(0)|}\right) > \frac{1}{2}|R_q(0)|. \tag{59}$$

We now use the trivial inequalities

$$3^{-n} \leqslant \left(\frac{1-|q|}{1+|q|}\right)^n \leqslant |\Gamma_q(n+1)| \leqslant \left(\frac{1+|q|}{1-|q|}\right)^n \leqslant 3^n, \qquad n = 0,1,2,\dots,$$

for the estimate of all the Γ_q-factors in $R_q(0)$. Since

$$(b_2-a_2-1)+(b_3-a_3-1)+(a_1-b_1)+(a_1+a_2+a_3-3)+(b_1+b_2+b_3-3) < 3(b_2+b_3)-1,$$

this leads to the estimates

$$3^{-3(b_2+b_3)+1} < |R_q(0)| < 3^{3(b_2+b_3)-1}. \tag{60}$$

Combining inequalities (58)–(60) we obtain the required estimates (55). The proof of the proposition is complete.

Proposition 4. *The coefficient* $A = A_q(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{Q}(q)$ *in the representation* (52) *for* $|q| \leqslant 1/2$ *has the estimate*

$$|A| < 3^{2(b_2+b_3)} \cdot |q|^{a_1(a_1-1)/2+a_2(a_2-1)/2+a_3(a_3-1)/2-(b_2-1)(b_3-1)}. \tag{61}$$

Proof. It will be useful to introduce an ordered version $(\boldsymbol{a}^*,\boldsymbol{b}^*)$ of the set of parameters $(\boldsymbol{a},\boldsymbol{b})$; namely,

$$b_1^* = 1 < a_1^* \leqslant a_2^* \leqslant a_3^* < b_2^* \leqslant b_3^*,$$
$$\{a_1^*, a_2^*, a_3^*\} = \{a_1,a_2,a_3\}, \quad \{b_1^*, b_2^*, b_3^*\} = \{b_1,b_2,b_3\}.$$

By functional equation (56) we obtain

$$\frac{\Gamma_q(t+a_j)}{\Gamma_q(t+b_j)} = \begin{cases} \dfrac{(1-q^{t+b_j})(1-q^{t+b_j+1})\cdots(1-q^{t+a_j-1})}{(1-q)^{a_j-b_j}} & \text{if } j=1, \\ \dfrac{(1-q)^{b_j-a_j}}{(1-q^{t+a_j})(1-q^{t+a_j+1})\cdots(1-q^{t+b_j-1})} & \text{if } j=2,3; \end{cases}$$

thus, $R_q(t)$ in (53) is a rational function of the variable $T=q^t$ over $\mathbb{Q}(q)=\mathbb{Q}(q^{-1})$:

$$\begin{aligned} R_q(t) &= \frac{[b_2-a_2-1]_q!\,[b_3-a_3-1]_q!}{[a_1-b_1]_q!} \cdot \frac{(1-q^{b_1}T)\cdots(1-q^{a_1-1}T)}{(1-q)^{a_1-b_1}} \\ &\quad\times \frac{(1-q)^{b_2-a_2-1}}{(1-q^{a_2}T)\cdots(1-q^{b_2-1}T)} \cdot \frac{(1-q)^{b_3-a_3-1}}{(1-q^{a_3}T)\cdots(1-q^{b_3-1}T)} \\ &\quad\times T^{b_2+b_3-a_1-a_2-a_3-1}. \end{aligned} \tag{62}$$

Since the degree of the denominator of (62) regarded as a function of T is greater by 2 than the degree of its numerator, it follows that

$$R_q(t) = O(T^{-2}) \qquad \text{as } T \to \infty; \tag{63}$$

so that $R_q(t)$ can be expanded in a sum of partial fractions:

$$R_q(t) = \sum_{k=a_3^*}^{b_2^*-1} \frac{A_k}{(1-q^kT)^2} + \sum_{k=a_2^*}^{b_3^*-1} \frac{B_k}{1-q^kT}.$$

Condition (63) yields

$$\sum_{k=a_2^*}^{b_3^*-1} B_k q^{-k} = -\sum_{k=a_2^*}^{b_3^*-1} \operatorname{Res}_{T=q^{-k}} R_q(t) = \operatorname{Res}_{T=\infty} R_q(t) = 0;$$

consequently,

$$\begin{aligned} G_q(\boldsymbol{a},\boldsymbol{b}) &= \sum_{t=0}^{\infty}\biggl(\sum_{k=a_3^*}^{b_2^*-1} \frac{A_k q^t}{(1-q^{t+k})^2} + \sum_{k=a_2^*}^{b_3^*-1} \frac{B_k q^t}{1-q^{t+k}}\biggr) \\ &= \sum_{k=a_3^*}^{b_2^*-1} A_k q^{-k} \sum_{t=0}^{\infty} \frac{q^{t+k}}{(1-q^{t+k})^2} + \sum_{k=a_3^*}^{b_2^*-1} B_k q^{-k} \sum_{t=0}^{\infty} \frac{q^{t+k}}{1-q^{t+k}} \\ &= \sum_{k=a_3^*}^{b_2^*-1} A_k q^{-k} \biggl(\sum_{l=1}^{\infty} - \sum_{l=1}^{k-1}\biggr) \frac{q^l}{(1-q^l)^2} + \sum_{k=a_2^*}^{b_3^*-1} B_k q^{-k} \biggl(\sum_{l=1}^{\infty} - \sum_{l=1}^{k-1}\biggr) \frac{q^l}{1-q^l} \\ &= A\zeta_q(2) - B, \end{aligned}$$

where

$$A=\sum_{k=a_3^*}^{b_2^*-1} A_k q^{-k}, \tag{64}$$

$$B=\sum_{k=a_3^*}^{b_2^*-1} A_k q^{-k}\sum_{l=1}^{k-1}\frac{q^l}{(1-q^l)^2}+\sum_{k=a_2^*}^{b_3^*-1} B_k q^{-k}\sum_{l=1}^{k-1}\frac{q^l}{1-q^l}$$

are rational functions of the variable q. Using the representation (62) we obtain explicit formulae for the coefficients A_k, $a_3^*\leqslant k<b_2^*$ (recall that $b_1=1$):

$$\begin{aligned}
A_k&=R_q(t)(1-q^kT)^2\big|_{T=q^{-k}}=R_q(t)(1-q^{t+k})^2\big|_{t=-k}\\
&=(-1)^{a_1-b_1}q^{(a_1-b_1)(a_1+b_1-2k-1)/2}\begin{bmatrix}k-b_1\\a_1-b_1\end{bmatrix}_q\\
&\quad\times(-1)^{k-a_2}q^{(k-a_2)(k-a_2+1)/2}\begin{bmatrix}b_2-a_2-1\\k-a_2\end{bmatrix}_q\\
&\quad\times(-1)^{k-a_3}q^{(k-a_3)(k-a_3+1)/2}\begin{bmatrix}b_3-a_3-1\\k-a_3\end{bmatrix}_q\cdot q^{-k(b_2+b_3-a_1-a_2-a_3-1)}\\
&=(-1)^{a_1+a_2+a_3-1}q^{a_1(a_1-1)/2+a_2(a_2-1)/2+a_3(a_3-1)/2-k(b_2+b_3-3)+k^2}\\
&\quad\times\begin{bmatrix}k-b_1\\a_1-b_1\end{bmatrix}_q\begin{bmatrix}b_2-a_2-1\\k-a_2\end{bmatrix}_q\begin{bmatrix}b_3-a_3-1\\k-a_3\end{bmatrix}_q,\qquad a_3^*\leqslant k<b_2^*.
\end{aligned}$$

The function $k^2-k(b_2+b_3-2)$ decreases for k between a_3^* and $b_2^*-1=\min\{b_2,b_3\}-1$ and takes its minimum value $-(b_2-1)(b_3-1)$ for $k=b_2^*-1$. In addition,

$$\left|\begin{bmatrix}k-b_1\\a_1-b_1\end{bmatrix}_q\right|=\left|\frac{(1-q^{k-a_1+1})\cdots(1-q^{k-b_1})}{(1-q)(1-q^2)\cdots(1-q^{a_1-b_1})}\right|\leqslant\left(\frac{1+|q|}{1-|q|}\right)^{a_1-b_1},$$

$$\begin{aligned}
\left|\begin{bmatrix}b_j-a_j-1\\k-a_j\end{bmatrix}_q\right|&=\left|\frac{(1-q)(1-q^2)\cdots(1-q^{b_j-a_j-1})}{(1-q)\cdots(1-q^{b_j-k-1})\cdot(1-q)\cdots(1-q^{k-b_j})}\right|\\
&\leqslant\left(\frac{1+|q|}{1-|q|}\right)^{b_j-a_j-1},\qquad j=2,3.
\end{aligned}$$

Hence

$$\begin{aligned}
|A_kq^{-k}|\leqslant&\left(\frac{1+|q|}{1-|q|}\right)^{a_1-a_2-a_3+b_2+b_3-3}\\
&\times|q|^{a_1(a_1-1)/2+a_2(a_2-1)/2+a_3(a_3-1)/2-(b_2-1)(b_3-1)},\qquad a_3^*\leqslant k<b_2^*.
\end{aligned}$$

Finally, using the inequalities $a_1<b_2$ and

$$\frac{1+|q|}{1-|q|}\leqslant 3$$

for $|q|\leqslant 1/2$ and the fact that the sum in (64) contains at most $b_3<3^{b_3}$ terms we arrive at the required estimate (61). The proof of the proposition is complete.

§ 6. Irrationality measure for $\zeta_q(2)$

Finally, we fix a set of integer parameters (*directions*) $(\boldsymbol{\alpha},\boldsymbol{\beta})$ satisfying the conditions

$$\{\beta_1=0\}\leqslant\{\alpha_1,\alpha_2,\alpha_3\}\leqslant\{\beta_2,\beta_3\},\qquad \alpha_1+\alpha_2+\alpha_3\leqslant\beta_1+\beta_2+\beta_3, \tag{65}$$

and for each $n=0,1,2,\dots$ assign to this set starting data (9) as follows:

$$\begin{aligned} a_1&=\alpha_1n+1, & a_2&=\alpha_2n+1, & a_3&=\alpha_3n+1,\\ b_1&=\beta_1n+1, & b_2&=\beta_2n+2, & b_3&=\beta_3n+2. \end{aligned} \tag{66}$$

Then defining a set of auxiliary parameters $\boldsymbol{c}$ by the rule

$$\begin{aligned} c_{00}&=(\beta_1+\beta_2+\beta_3)-(\alpha_1+\alpha_2+\alpha_3),\\ c_{jk}&=\begin{cases}\alpha_j-\beta_k & \text{for } k=1,\\ \beta_k-\alpha_j & \text{for } k=2,3,\end{cases} \qquad j,k=1,2,3, \end{aligned} \tag{67}$$

we see that the 10-element set $\boldsymbol{c}\cdot n$ corresponds to the set of parameters (66) as prescribed by (16). We associate with the set (67) the previously defined characteristics $m(\boldsymbol{c})$, $m_1^*(\boldsymbol{c})$, $m_2^*(\boldsymbol{c})$, $M(\boldsymbol{c})$ and consider the function

$$\begin{aligned} \omega_0(z)=\max_{0\leqslant k\leqslant 11}\bigl(&\lfloor c_{00}\cdot z\rfloor+\lfloor c_{21}\cdot z\rfloor+\lfloor c_{22}\cdot z\rfloor+\lfloor c_{33}\cdot z\rfloor+\lfloor c_{31}\cdot z\rfloor\\ &-\lfloor \mathfrak{g}_kc_{00}\cdot z\rfloor-\lfloor \mathfrak{g}_kc_{21}\cdot z\rfloor-\lfloor \mathfrak{g}_kc_{22}\cdot z\rfloor-\lfloor \mathfrak{g}_kc_{33}\cdot z\rfloor-\lfloor \mathfrak{g}_kc_{31}\cdot z\rfloor\bigr), \end{aligned} \tag{68}$$

where the representatives $\mathfrak{g}_k$, $k=0,1,\dots,11$, of the left cosets of the quotient $\mathfrak{G}/\mathfrak{G}_0$ and their action on the parameters $c_{00},c_{21},c_{22},c_{33},c_{31}$ are defined in (50) and (51). Note that the $\mathfrak{G}$-stability of the characteristic $m(\boldsymbol{c})$ yields the 1-periodicity of the function (68).

Proposition 5. *In the above notation let*

$$C_0=M-\frac{3}{\pi^2}\biggl({m_1^*}^2+{m_2^*}^2+\int_0^1\omega_0(z)\,\mathrm{d}\psi'(z)\biggr),\qquad C_1=\beta_2\beta_3-\frac{\alpha_1^2+\alpha_2^2+\alpha_3^2}{2}.$$

If $C_0>0$, then $\zeta_q(2)$ is an irrational number for each $q=1/p$ with $p\in\mathbb{Z}\setminus\{0,\pm1\}$, and the estimate

$$\mu(\zeta_q(2))\leqslant\frac{C_1}{C_0} \tag{69}$$

holds for the irrationality exponent.

Proof. Let $q^{-1}=p\in\mathbb{Z}\setminus\{0,\pm1\}$. For each set of directions $(\boldsymbol{\alpha},\boldsymbol{\beta})$ and the corresponding set (67) consider the sequences

$$H_n:=H(\boldsymbol{c}n),\qquad L_n:=p^{-Mn^2}\cdot D_{m_1^*n}(p)D_{m_2^*n}(p)\cdot\prod_{l=1}^{m_1^*n}\Phi_l^{-\omega_0(n/l)}(p),$$
$$n=0,1,2,\dots.$$

Since $M(\boldsymbol{c}n) = Mn^2$, $m_1^*(\boldsymbol{c}n) = m_1^*n$, $m_2^*(\boldsymbol{c}n) = m_2^*n$, and $\nu_l = \omega_0(n/l)$ by (49), $n = 0, 1, 2, \dots$, Proposition 2 ensures the inclusions

$$\widetilde{H}_n := L_n H_n \in \mathbb{Z}[p]\zeta_q(2) + \mathbb{Z}[p] \subset \mathbb{Z}\zeta_q(2) + \mathbb{Z}, \qquad n = 0, 1, 2, \dots .$$

On the other hand writing the linear forms H_n as $H_n = A_n\zeta_q(2) - B_n$, $n = 0, 1, 2, \dots$, and applying Propositions 3 and 4 we conclude that

$$\lim_{n\to\infty} \frac{\log|H_n|}{n^2} = 0 \qquad \text{and} \qquad \varlimsup_{n\to\infty} \frac{\log|A_n|}{n^2} \leqslant C_1 \log|p|, \tag{70}$$

while the asymptotic behaviour of the sequence L_n, $n = 0, 1, 2, \dots$, is determined by Lemmas 1 and 2:

$$\lim_{n\to\infty} \frac{\log|L_n|}{n^2} = -C_0 \log|p|. \tag{71}$$

Hence if $C_0 > 0$, then the irrationality of $\zeta_q(2)$ follows from the estimates

$$0 < |\widetilde{H}_n| < |p|^{-(C_0-\varepsilon)n^2},$$

which hold for all $n \geqslant n_0(\varepsilon)$, where one can take $C_0/2$ for $\varepsilon > 0$. Inequality (69) follows from limit relations (70) and (71) in the standard way (see, for instance, [19], § 11.3, Exercise 3 or [20], Lemma 2). This completes the proof of the proposition.

Proof of the theorem. Examining one after another all integer directions $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfying conditions (65) and $\beta_2 + \beta_3 \leqslant 100$ by means of a special program for the `PARI-GP` calculator we have discovered that the best estimate (3) for the irrationality exponent of $\zeta_q(2)$ is achieved (up to an action of the group $\mathfrak{G}$ and multiplication of the direction vector by a positive integer) on the following set:

$$\alpha_1 = 5, \quad \alpha_2 = 6, \quad \alpha_3 = 7, \qquad \beta_2 = 14, \quad \beta_3 = 15.$$

In this case $M = 74$, $m_1^* = 11$, $m_2^* = 10$, and

$$\omega_0(z) = \begin{cases} 0 & \text{if } z \in \left[0, \frac{1}{11}\right) \cup \left[\frac{1}{9}, \frac{1}{8}\right) \cup \left[\frac{2}{9}, \frac{1}{4}\right) \cup \left[\frac{1}{3}, \frac{4}{11}\right) \cup \left[\frac{4}{9}, \frac{1}{2}\right) \\ & \qquad \cup \left[\frac{3}{5}, \frac{5}{8}\right) \cup \left[\frac{7}{10}, \frac{8}{11}\right) \cup \left[\frac{4}{5}, \frac{5}{6}\right), \\ 1 & \text{if } z \in \left[\frac{1}{11}, \frac{1}{9}\right) \cup \left[\frac{1}{8}, \frac{2}{11}\right) \cup \left[\frac{1}{5}, \frac{2}{9}\right) \cup \left[\frac{1}{4}, \frac{1}{3}\right) \cup \left[\frac{4}{11}, \frac{3}{8}\right) \\ & \qquad \cup \left[\frac{2}{5}, \frac{4}{9}\right) \cup \left[\frac{1}{2}, \frac{6}{11}\right) \cup \left[\frac{5}{9}, \frac{3}{5}\right) \cup \left[\frac{5}{8}, \frac{7}{10}\right) \cup \left[\frac{8}{11}, \frac{3}{4}\right) \\ & \qquad \cup \left[\frac{7}{9}, \frac{4}{5}\right) \cup \left[\frac{5}{6}, \frac{7}{8}\right) \cup \left[\frac{8}{9}, \frac{9}{10}\right), \\ 2 & \text{if } z \in \left[\frac{2}{11}, \frac{1}{5}\right) \cup \left[\frac{3}{8}, \frac{2}{5}\right) \cup \left[\frac{6}{11}, \frac{5}{9}\right) \cup \left[\frac{3}{4}, \frac{7}{9}\right) \cup \left[\frac{7}{8}, \frac{8}{9}\right) \end{cases}$$

for $z \in [0, 1)$. Hence

$$C_0 = 74 - \frac{3}{\pi^2}(11^2 + 10^2 - 102.57252091\dots) = 38.00236293\dots,$$
$$C_1 = 14 \cdot 15 - \frac{5^2 + 6^2 + 7^2}{2} = 155,$$

and we arrive at the required estimate (3) by Proposition 5. The proof is complete.

§ 7. A q-analogue of the Apéry sequence

The choice of the directions

$$a_1 = a_2 = a_3 = n+1, \quad b_1 = 1, \quad b_2 = b_3 = 2n+2, \qquad \text{where} \quad n = 0, 1, 2, \dots, \tag{72}$$

leads to the quantities $C_0 = 1 - 6/\pi^2 > 0$ and $C_1 = 5/2$ in the notation of Proposition 5, and hence to the irrationality of $\zeta_q(2)$ for $q^{-1} \in \mathbb{Z} \setminus \{0, \pm 1\}$. The corresponding estimate for the irrationality exponent in this case takes the following form:

$$\mu(\zeta_q(2)) \leqslant \frac{5\pi^2}{2\pi^2 - 12} = 6.37636524\dots .$$

The aim of this section is to demonstrate that the case (72) is a precise q-extension of Apéry's original proof of the irrationality of $\zeta(2)$.

We fix an integer $n \geqslant 0$ and write the partial-fraction expansion of the rational function (53) with respect to the variable $T = q^t$:

$$\begin{aligned}
R_q(t) &= \frac{(1-qT)\cdots(1-q^nT)}{(1-q^{n+1}T)\cdots(1-q^{2n+1}T)} \cdot \frac{(q;q)_n T^n}{(1-q^{n+1}T)\cdots(1-q^{2n+1}T)} \\
&= (-1)^n \sum_{k=0}^{n} \begin{bmatrix} k+n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1)^k q^{k(k+1)/2 - kn - n(n+1)/2}}{1 - q^{k+n+1}T} \\
&\qquad \times \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{(-1)^j q^{j(j+1)/2 - jn - n(n+1)}}{1 - q^{j+n+1}T} \\
&= (-1)^n \sum_{k=0}^{n} \begin{bmatrix} k+n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q^2 \frac{q^{k(k+1) - 2kn - 3n(n+1)/2}}{(1 - q^{t+k+n+1})^2} \\
&\qquad + (-1)^n \sum_{k=0}^{n} \sum_{\substack{j=0 \\ j \ne k}}^{n} \begin{bmatrix} k+n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q q^{k(k+1)/2 + j(j+1)/2 - (k+j)n - 3n(n+1)/2} \\
&\qquad \times \frac{(-1)^{k+j}}{q^k - q^j} \biggl(\frac{1}{1 - q^{t+k+n+1}} - \frac{1}{1 - q^{t+j+n+1}} \biggr).
\end{aligned} \tag{73}$$

Bearing in mind the equalities

$$R_q(t) = 0 \qquad \text{for } t = -1, -2, \dots, -n$$

and

$$\left.\begin{aligned}
\sum_{t=-n}^{\infty} \frac{q^t}{(1 - q^{t+k+n+1})^2} &= q^{-(k+n+1)} \biggl(\zeta_q(2) - \sum_{l=1}^{k} \frac{q^l}{(1-q^l)^2} \biggr), \\
\sum_{t=-n}^{\infty} \frac{q^t}{1 - q^{t+k+n+1}} &= q^{-(k+n+1)} \biggl(\zeta_q(1) - \sum_{l=1}^{k} \frac{q^l}{1-q^l} \biggr),
\end{aligned}\right\} \qquad k = 0, 1, \dots, n,$$

we obtain from (73) the linear form

$$\begin{aligned}
H_n(q) &:= (-1)^n q^{(3n+2)(n+1)/2} \sum_{t=0}^{\infty} R_q(t) = (-1)^n q^{(3n+2)(n+1)/2} \sum_{t=-n}^{\infty} R_q(t) \\
&= A_n(q)\zeta_q(2) - B_n(q)
\end{aligned} \tag{74}$$

(with the coefficient of $\zeta_q(1)$ in (74) equal to zero by Proposition 1), where

$$A_n(q)=\sum_{k=0}^{n}\begin{bmatrix}k+n\\k\end{bmatrix}_q\begin{bmatrix}n\\k\end{bmatrix}_q^2 q^{k^2-2kn},$$

$$B_n(q)=\sum_{k=0}^{n}\begin{bmatrix}k+n\\k\end{bmatrix}_q\begin{bmatrix}n\\k\end{bmatrix}_q^2 q^{k^2-2kn}\sum_{l=1}^{k}\frac{q^l}{(1-q^l)^2}$$
$$+\sum_{k=0}^{n}\sum_{\substack{j=0\\j\neq k}}^{n}\begin{bmatrix}k+n\\k\end{bmatrix}_q\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}n\\j\end{bmatrix}_q q^{k(k+1)/2+j(j+1)/2-(k+j)n}$$
$$\times\frac{(-1)^{k+j}}{q^k-q^j}\biggl(q^{-k}\sum_{l=1}^{k}\frac{q^l}{1-q^l}-q^{-j}\sum_{l=1}^{j}\frac{q^l}{1-q^l}\biggr).$$

Letting $q\to 1$ we now see that

$$A_n:=\lim_{q\to1}A_n(q)=\sum_{k=0}^{n}\binom{k+n}{k}\binom{n}{k}^2,$$

$$B_n:=\lim_{q\to1}(1-q)^2B_n(q)=\sum_{k=0}^{n}\binom{k+n}{k}\binom{n}{k}^2\sum_{l=1}^{k}\frac{1}{l^2}$$
$$+\sum_{k=0}^{n}\sum_{\substack{j=0\\j\neq k}}^{n}\binom{k+n}{k}\binom{n}{k}\binom{n}{j}\frac{(-1)^{k+j}}{j-k}\biggl(\sum_{l=1}^{k}\frac{1}{l}-\sum_{l=1}^{j}\frac{1}{l}\biggr).$$

It remains to observe (see, for instance, [21], § 4) that the sequence of linear forms

$$H_n:=A_n\zeta(2)-B_n,\qquad n=0,1,2,\dots,$$

is precisely the Apéry sequence [12].

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