APPROXIMATIONS TO q-LOGARITHMS AND q-DILOGARITHMS, WITH APPLICATIONS TO q-ZETA VALUES

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Abstract. We construct simultaneous rational approximations to the q-series $L_1(x_1;q)$ and $L_1(x_2;q)$, and, if $x=x_1=x_2$, to the series $L_1(x;q)$ and $L_2(x;q)$, where

$$L_1(x;q)=\sum_{n=1}^{\infty}\frac{(xq)^n}{1-q^n}=\sum_{n=1}^{\infty}\frac{xq^n}{1-xq^n}, \quad L_2(x;q)=\sum_{n=1}^{\infty}\frac{n(xq)^n}{1-q^n}=\sum_{n=1}^{\infty}\frac{xq^n}{(1-xq^n)^2}.$$

Applying the construction, we obtain quantitative linear independence over $\mathbb{Q}$ of the numbers in the following collections: 1, $\zeta_q(1)=L_1(1;q)$, $\zeta_{q^2}(1)$, and 1, $\zeta_q(1)$, $\zeta_q(2)=L_2(1;q)$ for $q=1/p$, $p\in\mathbb{Z}\setminus\{0,\pm1\}$.

Let q be a variable taking values in the disc $|q|<1$, and let $p=1/q$ be its reciprocal. The q-logarithm is defined by the series

$$\begin{aligned}L_1(x;q)=-\ln_q(1-x)&=\sum_{n=1}^{\infty}\frac{(xq)^n}{1-q^n}=\sum_{n=1}^{\infty}\frac{x^n}{p^n-1}\\&=\sum_{n=1}^{\infty}\frac{xq^n}{1-xq^n}=x\sum_{n=1}^{\infty}\frac{1}{p^n-x}, \qquad |x|<|p|=|q|^{-1}.\end{aligned}$$

It really inherits certain properties of the series

$$\mathrm{Li}_1(x)=\sum_{n=1}^{\infty}\frac{x^n}{n}, \qquad |x|<1,$$

for the ordinary logarithm $-\log(1-x)$, although from the number-theoretical point of view it is, in a sense, more intriguing: there are no transcendence results, at least for its values at rational points x. Even the irrationality is usually asserted for x rational but p integer, $|p|>1$. P. Erdős proved [Er] that the q-harmonic series $\zeta_q(1)=L_1(1;q)$ is irrational for $q=1/p$, $p\in\mathbb{Z}\setminus\{0,\pm1\}$, already in 1948, and only recently, based on new ideas, further results in this direction, first quantitative and then qualitative, were obtained in [Be], [Bo], [BV1], [As], [MVZ], [BZ].

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The series, which may be regarded as a possible q-extension of the series

$$\mathrm{Li}_2(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2}, \qquad |x|\leqslant 1,$$

for the dilogarithm, is

$$\begin{aligned} L_2(x;q)&=\sum_{n=1}^{\infty}\frac{n(xq)^n}{1-q^n}=\sum_{n=1}^{\infty}\frac{nx^n}{p^n-1} \\ &=\sum_{n=1}^{\infty}\frac{xq^n}{(1-xq^n)^2}=x\sum_{n=1}^{\infty}\frac{p^n}{(p^n-x)^2}, \qquad |x|<|p|=|q|^{-1}. \end{aligned}$$

In particular, the value $\zeta_q(2)=L_2(1;q)$ is in a very interesting parallel with the number $\zeta(2)$ (see [Zu2]). On the other hand, the equations

$$x\frac{\mathrm{d}}{\mathrm{d}x}L_1(x;q)=L_2(x;q), \qquad x\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Li}_2(x)=\mathrm{Li}_1(x)$$

show that our q-analogues of the logarithm and dilogarithm have an opposite differential relationship than their originals.

In Section 1, we present a general construction of simultaneous rational approximations to the q-functions $L_1(x_1;q)$ and $L_1(x_2;q)$, where x_1,x_2 are distinct fixed complex numbers. It happens so that the only case, when we are able to apply the functional construction for getting an arithmetic result for the values, is $x_1=1$ and $x_2=-1$. The following theorem and its corollary are proved, independently and by a completely different method, by P. Bundschuh and K. Väänänen in [BV2].

Theorem 1. *Let $q=1/p$ for some $p\in\mathbb{Z}\setminus\{0,\pm1\}$. Then the numbers*

$$1, \quad L_1(1;q)=\sum_{n=1}^{\infty}\frac{1}{p^n-1} \quad \text{and} \quad L_1(-1;q)=-\sum_{n=1}^{\infty}\frac{1}{p^n+1}$$

are linearly independent over $\mathbb{Q}$. Moreover, for any $\varepsilon>0$ there exists a positive constant $\mathcal{X}(\varepsilon)$ such that

$$|X_0+X_1L_1(1;q)+X_2L_1(-1;q)|\geqslant X^{-2(\pi^2+4)/(\pi^2-8)-\varepsilon}, \qquad X=\max\{|X_1|,|X_2|\}, \tag{1}$$

for any integers X_0,X_1,X_2 satisfying $X\geqslant\mathcal{X}(\varepsilon)$. (Numerically, $2(\pi^2+4)/(\pi^2-8)=14.83694025\dots$.)

Since

$$L_1(1;q)=\zeta_q(1) \quad \text{and} \quad \frac{1}{2}\big(L_1(1;q)+L_1(-1;q)\big)=\zeta_{q^2}(1),$$

we have the following curious

Corollary. *Let $q=1/p$ for some $p\in\mathbb{Z}\setminus\{0,\pm1\}$. Then the numbers*

$$1, \quad \zeta_q(1)=\sum_{n=1}^{\infty}\frac{1}{p^n-1} \quad \text{and} \quad \zeta_{q^2}(1)=\sum_{n=1}^{\infty}\frac{1}{p^{2n}-1}$$

are linearly independent over $\mathbb{Q}$. Moreover, for any $\varepsilon > 0$ there exists a positive constant $\mathcal{X}(\varepsilon)$ such that

$$|X_0 + X_1\zeta_q(1) + X_2\zeta_{q^2}(1)| \geqslant X^{-2(\pi^2+4)/(\pi^2-8)-\varepsilon}, \qquad X = \max\{|X_1|, |X_2|\},$$

for any integers X_0, X_1, X_2 satisfying $X \geqslant \mathcal{X}(\varepsilon)$.

If $x_1 = x_2 = x$, our construction may be developed further to provide simultaneous rational approximations to the q-functions $L_1(x;q)$ and $L_2(x;q)$. This is done in Section 2 and the only arithmetic application for the values is the following result, previously obtained by K. Postelmans and W. Van Assche in [PA] using multiple (in fact, double) little q-Jacobi polynomials.

Theorem 2. *Let $q = 1/p$ for some $p \in \mathbb{Z} \setminus \{0, \pm 1\}$. Then the numbers*

$$1, \quad \zeta_q(1) = L_1(1;q) = \sum_{n=1}^{\infty} \frac{1}{p^n - 1} \quad \text{and} \quad \zeta_q(2) = L_2(1;q) = \sum_{n=1}^{\infty} \frac{p^n}{(p^n-1)^2}$$

are linearly independent over $\mathbb{Q}$. Moreover, for any $\varepsilon > 0$ there exists a positive constant $\mathcal{X}(\varepsilon)$ such that

$$|X_0 + X_1\zeta_q(1) + X_2\zeta_q(2)| \geqslant X^{-2(\pi^2+4)/(\pi^2-8)-\varepsilon}, \qquad X = \max\{|X_1|, |X_2|\}, \tag{2}$$

for any integers X_0, X_1, X_2 satisfying $X \geqslant \mathcal{X}(\varepsilon)$.

Recall standard q-notations, which will be used throughout the paper:

$$(a;q)_n = \prod_{\nu=1}^{n} (1 - aq^{\nu-1}), \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k \cdot (q;q)_{n-k}} \in \mathbb{Z}[q],$$

where $k = 0, 1, \dots, n$ and $n = 0, 1, 2, \dots$. The definition of the q-hypergeometric series appears later in (15).

1. Simultaneous approximations to two q-logarithms

1.1. Let x_1, x_2 be distinct complex arguments of two q-logarithms. The construction below will depend on the positive integers n and m satisfying $m \geqslant 2n$. Take

$$\widetilde{R}(T) = \frac{(qT/x_1;q)_n (qT/x_2;q)_n}{(q^{n+1}T;q)_{2n+1}} = \frac{\prod_{j=1}^{n}(1 - q^jT/x_1)(1 - q^jT/x_2)}{\prod_{k=0}^{2n}(1 - q^{k+n+1}T)}. \tag{3}$$

Lemma 1. *The following partial fraction decomposition is valid:*

$$\widetilde{R}(T) = \sum_{k=0}^{2n} \frac{A_k}{1 - q^{k+n+1}T}, \tag{4}$$

where

$$A_k = (-1)^k (x_1x_2)^{-n} q^{-n(n+1)-2kn+k(k+1)/2} \frac{(q^{k+1}x_1;q)_n (q^{k+1}x_2;q)_n}{(q;q)_k (q;q)_{2n-k}} \tag{5}$$

$$= (x_1x_2)^{-n} q^{-n(n+1)} \frac{(qx_1;q)_n (qx_2;q)_n}{(q;q)_{2n}} \cdot \frac{(q^{-2n};q)_k (q^{n+1}x_1;q)_k (q^{n+1}x_2;q)_k}{(q;q)_k (qx_1;q)_k (qx_2;q)_k} q^k, \tag{6}$$

and also

$$A_k p^{(k+n+1)(m+1)} = p^{M+k(k+1)/2+k(m-2n)} \frac{(p^{k+1}/x_1;p)_n (p^{k+1}/x_2;p)_n}{(p;p)_k (p;p)_{2n-k}} \tag{7}$$

with $M = n(m+2n+2)+m+1$.

Proof. The existence and uniqueness of the decomposition (4) is a classical knowledge combined with the fact that $\widetilde{R}(T) = O(T^{-1})$ as $T \to \infty$. Moreover, we may use the standard procedure for determining the unknown coefficients:

$$A_k = \widetilde{R}(T)(1-q^{k+n+1}T)\big|_{T=q^{-(k+n+1)}} = \frac{(q^{-(k+n)}/x_1;q)_n (q^{-(k+n)}/x_2;q)_n}{(q^{-k};q)_k (q;q)_{2n-k}}.$$

The latter expression implies formulae (5) and (6). Then we deduce

$$A_k p^{(k+n+1)(m+1)} = p^{(k+n+1)(m+1)+(2n-k)(2n-k+1)/2} \frac{(p^{k+1}/x_1;p)_n (p^{k+1}/x_2;p)_n}{(p;p)_k (p;p)_{2n-k}}$$

that after simple reduction becomes (7). $\square$

1.2. Now let $R(T) = \widetilde{R}(T) \cdot T^{m+1}$, and let x denote any of the two numbers x_1 or x_2. Note that the function $R(T)$ has zeros at $T = xq^t$ for $t = -1, -2, \dots, -n$. Consider the quantities

$$I(x) = \sum_{t=0}^{\infty} R(T)\big|_{T=xq^t} = \sum_{t=-n}^{\infty} R(T)\big|_{T=xq^t}, \qquad x \in \{x_1, x_2\}. \tag{8}$$

Lemma 2. *We have*

$$I(x_j) = AL_1(x_j;q) - A^*(x_j) - A^{**}(x_j), \qquad j = 1, 2, \tag{9}$$

where

$$A = \sum_{k=0}^{2n} A_k p^{(k+n+1)(m+1)}, \qquad A^*(x) = \sum_{k=0}^{2n} A_k p^{(k+n+1)(m+1)} \sum_{l=1}^{k} \frac{x}{p^l - x}, \tag{10}$$

$$A^{**}(x) = p^{(n+1)(m+1)} \sum_{l=0}^{m-1} \frac{x^{m-l}}{p^{m-l}-1} \sum_{k=0}^{2n} A_k q^{-k(l+1)} \tag{11}$$

In other words, $I(x_1)$ *and* $I(x_2)$ *viewed as functions of* $p = 1/q$ *realize simultaneous rational approximations to the q-logarithms* $L_1(x_1;q)$ *and* $L_1(x_2;q)$.

Proof. Write

$$\begin{aligned} I(x) &= \sum_{t=-n}^{\infty} R(T)\big|_{T=xq^t} = \sum_{t=-n}^{\infty} x^{m+1} q^{t(m+1)} \sum_{k=0}^{2n} \frac{A_k}{1-xq^{k+n+t+1}} \\ &= x^{m+1} \sum_{k=0}^{2n} A_k q^{-(k+n+1)(m+1)} \sum_{t=-n}^{\infty} \frac{q^{(k+n+t+1)(m+1)}}{1-xq^{k+n+t+1}} \\ &= x^{m+1} \sum_{k=0}^{2n} A_k q^{-(k+n+1)(m+1)} \sum_{l=k+1}^{\infty} \frac{q^{l(m+1)}}{1-xq^l}. \end{aligned}$$

Since

$$\begin{aligned}
x^{m+1}\sum_{l=k+1}^{\infty}\frac{q^{l(m+1)}}{1-xq^l}&=\sum_{l=k+1}^{\infty}\frac{xq^l}{1-xq^l}-\sum_{l=k+1}^{\infty}\frac{xq^l-x^{m+1}q^{l(m+1)}}{1-xq^l}\\
&=\sum_{l=1}^{\infty}\frac{xq^l}{1-xq^l}-\sum_{l=1}^{k}\frac{xq^l}{1-xq^l}-\sum_{l=k+1}^{\infty}\sum_{j=1}^{m}(xq^l)^j\\
&=L_1(x;q)-\sum_{l=1}^{k}\frac{xq^l}{1-xq^l}-\sum_{j=1}^{m}x^j\sum_{l=k+1}^{\infty}(q^j)^l\\
&=L_1(x;q)-\sum_{l=1}^{k}\frac{xq^l}{1-xq^l}-\sum_{j=1}^{m}\frac{x^jq^{j(k+1)}}{1-q^j},
\end{aligned}\tag{12}$$

we obtain

$$\begin{aligned}
I(x)&=\sum_{k=0}^{2n}A_kq^{-(k+n+1)(m+1)}\cdot L_1(x;q)-\sum_{k=0}^{2n}A_kq^{-(k+n+1)(m+1)}\sum_{l=1}^{k}\frac{xq^l}{1-xq^l}\\
&\quad-\sum_{k=0}^{2n}A_kq^{-(k+n+1)(m+1)}\sum_{j=1}^{m}\frac{x^jq^{j(k+1)}}{1-q^j}\\
&=\sum_{k=0}^{2n}A_kp^{(k+n+1)(m+1)}\cdot L_1(x;q)-\sum_{k=0}^{2n}A_kp^{(k+n+1)(m+1)}\sum_{l=1}^{k}\frac{x}{p^l-x}\\
&\quad-p^{(n+1)(m+1)}\sum_{j=1}^{m}\frac{(xq)^j}{1-q^j}\sum_{k=0}^{2n}A_kq^{-k(m+1-j)},
\end{aligned}$$

from which the result follows. Note that from (10), (11) and the explicit formulae for A_k, presented in Lemma 1, the quantities A, $A^*(x)$ and $A^{**}(x)$ are indeed rational functions of the variable $p=1/q$. $\square$

Remark. The above construction might be easily generalized: the s quantities

$$I(x_j)=\sum_{t=0}^{\infty}\frac{(qT/x_1;q)_n\cdots(qT/x_s;q)_n}{(q^{n+1}T;q)_{m+1}}T^{m+1}\bigg|_{T=x_jq^t},\qquad j=1,\dots,s,$$

correspond to (functional) simultaneous approximations to $L_1(x_1;q),\dots,L_1(x_s;q)$. A problem here consists in the fact that no arithmetic applications to the values are available if $s>2$.

1.3. From Lemma 1, multiplication of every A_k by $X^n(p;p)_{2n}$, where X is the product of the numerators of the rational numbers x_1 and x_2, and by a 'suitable' power of the polynomial p, gives us polynomials in $\mathbb{Z}[p]$, whence from (7) and the starting condition $m\geqslant 2n$ we deduce the following result.

Lemma 3. *We have*

$$X^np^{-M}(p;p)_{2n}\cdot A\in\mathbb{Z}[p],\qquad X^np^{-M}(p;p)_{2n}D_{2n}(p,x)\cdot A^*(x)\in\mathbb{Z}[p],\tag{13}$$

$$X^np^{-(n+1)(m+1)}(p;p)_{2n}D_m(p,1)\cdot A^{**}(x)\in\mathbb{Z}[p],\tag{14}$$

where $M = n(m+2n+2)+m+1$ and $D_N(p,x)$, $x \in \mathbb{C}$, denotes the least common multiple of the polynomials $p-x, p^2-x, \dots, p^N-x$ in the ring $\mathbb{Z}[p]$.

Inclusion (14) may be considerably improved by the application of the following q-hypergeometric identity.

Lemma 4. *For $s \geqslant 1$,*

$$
{}_{s+1}\phi_s\left(\begin{matrix} a, b_1, \dots, b_s \\ c_1, \dots, c_s \end{matrix} \;\middle|\; q, z\right) = \sum_{k=0}^{\infty} \frac{(a;q)_k (b_1;q)_k \cdots (b_s;q)_k}{(q;q)_k (c_1;q)_k \cdots (c_s;q)_k} z^k
= \frac{(az;q)_\infty}{(z;q)_\infty} \prod_{j=1}^{s} \frac{(b_j;q)_\infty}{(c_j;q)_\infty} \sum_{k_j=0}^{\infty} \frac{(c_j/b_j;q)_{k_j}}{(q;q)_{k_j}} b_j^{k_j} \frac{(z;q)_{k_1+\dots+k_s}}{(az;q)_{k_1+\dots+k_s}}. \tag{15}
$$

We do not reproduce the proof of this simple fact, since it follows lines of the proof in [GR], Section 1.4, of classical Heine's transform (corresponding to the case $s=1$).

The promised improvement of (14) is as follows.

Lemma 5. *We have*

$$
X^n p^{-M} D_m(p,1) \cdot A^{**}(x) \in \mathbb{Z}[p]. \tag{16}
$$

Proof. Using (6) and Lemma 4 with $s=2$ we see that

$$
\begin{aligned}
\sum_{k=0}^{2n} A_k q^{-k(l+1)} &= (x_1x_2)^{-n} q^{-n(n+1)} \frac{(qx_1;q)_n (qx_2;q)_n}{(q;q)_{2n}} \\
&\quad \times {}_3\phi_2\left(\begin{matrix} q^{-2n}, q^{n+1}x_1, q^{n+1}x_2 \\ qx_1, \quad qx_2 \end{matrix} \;\middle|\; q, q^{-l}\right) \\
&= (x_1x_2)^{-n} q^{-n(n+1)} \frac{(q^{-2n-l};q)_{2n}}{(q;q)_{2n}} \\
&\quad \times \sum_{k_1=0}^{n} \frac{(q^{-n};q)_{k_1}}{(q;q)_{k_1}} (q^{n+1}x_1)^{k_1} \sum_{k_2=0}^{n} \frac{(q^{-n};q)_{k_2}}{(q;q)_{k_2}} (q^{n+1}x_2)^{k_2} \\
&\quad \times \frac{(q^{-l};q)_{k_1+k_2}}{(q^{-2n-l};q)_{k_1+k_2}} \\
&= p^{n(n+1)+n(2n+1)} \frac{(p^{l+1};p)_{2n}}{(p;p)_{2n}} \sum_{k_1=0}^{n} \sum_{k_2=0}^{n} \frac{(p^{l-k_1-k_2+1};p)_{k_1+k_2}}{(p^{2n+l-k_1-k_2+1};p)_{k_1+k_2}} \\
&\quad \times \frac{(p^{n-k_1+1};p)_{k_1} (p^{n-k_2+1};p)_{k_2}}{(p;p)_{k_1} (p;p)_{k_2}} (-1)^{k_1+k_2} x_1^{k_1-n} x_2^{k_2-n} \\
&\quad \times p^{k_1(k_1+1)/2 + k_2(k_2+1)/2 - (k_1+k_2)(n+1)} \\
&= p^{n(2n+1)} \sum_{\substack{k_1=0 \\ k_1+k_2 \leqslant l}}^{n} \sum_{k_2=0}^{n} \begin{bmatrix} 2n+l-k_1-k_2 \\ 2n \end{bmatrix}_p \begin{bmatrix} n \\ k_1 \end{bmatrix}_p \begin{bmatrix} n \\ k_2 \end{bmatrix}_p (-1)^{k_1+k_2} \\
&\quad \times x_1^{k_1-n} x_2^{k_2-n} p^{(n-k_1)(n-k_1+1)/2 + (n-k_2)(n-k_2+1)/2}.
\end{aligned} \tag{17}
$$

Substituting this result into (11) we get the desired inclusion (16). $\square$

1.4. The asymptotic evaluation of the approximations $I^{(n)}(x_j) = I(x_j)$, $j = 1, 2$, and coefficients $A^{(n)} = A$ as $n \to \infty$ follows a standard scheme.

Lemma 6. *For $p \in \mathbb{Z} \setminus \{0, \pm 1\}$, we have*

$$\lim_{n\to\infty} \frac{\log |I^{(n)}(x_j)|}{n^2 \log |p|} = 0, \qquad j = 1, 2. \tag{18}$$

Proof. From (8) and $m \geqslant 2n$ we deduce that

$$I(x) = R(T)|_{T=x} + O(q^{m+1}) \sim \widetilde{R}(x) \qquad \text{as} \quad n \to \infty$$

for $x \in \{x_1, x_2\}$, with the immediate consequence (see (3))

$$\lim_{n\to\infty} \frac{\log |I^{(n)}(x_j)|}{n^2 \log |q|} = 0, \qquad j = 1, 2,$$

yielding (18). $\square$

Lemma 7. *Let $m = \lfloor \alpha n \rfloor$ with some real $\alpha \geqslant 2$ (the brackets $\lfloor \cdot \rfloor$ denote the integer part of a number). Then, for $p \in \mathbb{Z} \setminus \{0, \pm 1\}$, the following limit relation is valid:*

$$\lim_{n\to\infty} \frac{\log |A^{(n)}|}{n^2 \log |p|} = 3(1 + \alpha). \tag{19}$$

Proof. For the sequence $A = A^{(n)}$, we use the explicit formulae (10) and (5). We have $A = \sum_{k=0}^{2n} \widetilde{A}_k$, where

$$\begin{aligned} \widetilde{A}_k &= A_k p^{(k+n+1)(m+1)} \\ &= (-1)^k (x_1 x_2)^{-n} p^{(n+1)(n+m+1)+(2n+m+1)k-k(k+1)/2} \frac{(q^{k+1}x_1; q)_n (q^{k+1}x_2; q)_n}{(q; q)_k (q; q)_{2n-k}}. \end{aligned} \tag{20}$$

Since

$$\frac{\widetilde{A}_k}{\widetilde{A}_{k-1}} = -p^{2n+m-k+1} \frac{(1 - q^{k+n}x_1)(1 - q^{k+n}x_2)(1 - q^{2n-k+1})}{(1 - q^k x_1)(1 - q^k x_2)(1 - q^k)}, \qquad k = 1, 2, \dots, 2n,$$

we obtain $|\widetilde{A}_k| \geqslant |p| \cdot |\widetilde{A}_{k-1}|$ for $k = 1, 2, \dots, 2n$, unless n is sufficiently large. The latter inequalities give us

$$\begin{aligned} |A| &= \left| \sum_{k=0}^{2n} \widetilde{A}_k \right| \leqslant |\widetilde{A}_{2n}| \sum_{k=0}^{2n} |p|^{-k} < |\widetilde{A}_{2n}| \cdot \frac{1}{1 - |p|^{-1}}, \\ |A| &= \left| \sum_{k=0}^{2n} \widetilde{A}_k \right| \geqslant |\widetilde{A}_{2n}| - |\widetilde{A}_{2n-1}| \geqslant |\widetilde{A}_{2n}| \cdot \left(1 - \frac{1}{|p|}\right). \end{aligned} \tag{21}$$

Finally, by (20)

$$|\widetilde{A}_{2n}| = |x_1 x_2|^{-n} |p|^{3n^2+(3n+1)(m+1)} \big(1 + O(|q|)\big) \qquad \text{as} \quad n \to \infty,$$

that in combination with (21) and $m = \lfloor \alpha n \rfloor$ yield (19). $\square$

1.5. In order to prove a linear independence result for the values of $L_1(x;q)$ at two different points x_1, x_2, we should deal with the approximations having integer coefficients. This means that we are required to multiply our approximations (9) by

$$X^n p^{-M} \cdot \text{l.c.m.}\big((p;p)_{2n}, D_m(p,1)\big) \cdot \text{l.c.m.}\big(D_{2n}(p,x_1), D_{2n}(p,x_2)\big) \tag{22}$$

(see Lemmas 3, 5). Unfortunately, in spite of the high negative power of p, the factor (22) always increases to infinity with n and this fact, in view of Lemma 6, means that no arithmetic result could follow.

Nevertheless, in the special case $x_1 = 1$, $x_2 = -1$ we may improve inclusions (13) of Lemma 3 and use the precise estimates from [As], [MVZ] for the degree of the polynomials

$$\begin{aligned} \widehat{D}_{m,2n}(p) &= \text{l.c.m.}\big(D_{2n}(p,1), D_{2n}(p,-1), D_m(p,1)\big) \\ &= \text{l.c.m.}\big(D_m(p,1), D_{2n}(p,-1)\big), \qquad n = 1, 2, \dots, \end{aligned}$$

to get our Theorem 1.

Lemma 8. *If $x_1 = 1$ and $x_2 = -1$, then the following inclusions hold:*

$$p^{-M}(p;p^2)_n \cdot A \in \mathbb{Z}[p], \qquad p^{-M}(p;p^2)_n D_{2n}(p,\pm 1) \cdot A^*(\pm 1) \in \mathbb{Z}[p], \tag{23}$$

$$p^{-M} D_m(p,1) \cdot A^{**}(\pm 1) \in \mathbb{Z}[p]. \tag{24}$$

In other words, the factor

$$p^{-M}(p;p^2)_n \widehat{D}_{m,2n}(p)$$

is a common denominator of the approximations $I(1)$ and $I(-1)$.

Proof. The inclusion (24) is already shown in Lemma 5. By (7),

$$\begin{aligned} &(p;p^2)_n \cdot A_k p^{(k+n+1)(m+1)} \\ &\quad = p^{M+k(k+1)/2+k(m-2n)} \frac{(p;p)_{2n}}{(p^2;p^2)_n} \cdot \frac{(p^{k+1};p)_n(-p^{k+1};p)_n}{(p;p)_k(p;p)_{2n-k}} \\ &\quad = p^{M+k(k+1)/2+k(m-2n)} \begin{bmatrix} n+k \\ k \end{bmatrix}_{p^2} \begin{bmatrix} 2n \\ k \end{bmatrix}_p \in p^{(2n+1)^2}\mathbb{Z}[p]; \end{aligned}$$

thus, the inclusions (23) follow from (10). $\square$

Lemma 9 ([MVZ], Corollary of Lemma 1). *Suppose $2n \leqslant m \leqslant 4n$. Then*

$$\deg_p \widehat{D}_{m,2n}(p) = \frac{1}{\pi^2}(2m^2 + 4(2n)^2) + O(n \log n) \qquad \text{as} \quad n \to \infty.$$

In other words, for p integer, $|p| > 1$,

$$\lim_{n\to\infty} \frac{\log|\widehat{D}_{\lfloor \alpha n \rfloor, 2n}(p)|}{n^2 \log|p|} = \frac{2}{\pi^2}(8 + \alpha^2).$$

Finally, we present a non-vanishing property of the approximations $I^{(n)}(x)$, where $x \in \{x_1, x_2\} = \{\pm 1\}$.

Lemma 10. *For any $p\in\mathbb{Z}\setminus\{0,\pm1\}$ and any pair of rational numbers X_1 and X_2, $X_1^2+X_2^2\ne0$, we have*

$$X_1I^{(n)}(1)+X_2I^{(n)}(-1)\ne0$$

for all n sufficiently large.

Proof. Using the definition of $I(x)$ we deduce that

$$\begin{aligned}
I^{(n)}(\pm1)&=\sum_{t=0}^{\infty}R(T)\big|_{T=\pm q^t}=R(\pm1)+O(q^{m+1})\\
&=-\frac{(q;q)_n(-q;q)_n}{(\pm q^{n+1};q)_{2n+1}}+O(q^{2n+1})=-\frac{(q^2;q^2)_n}{(1\mp q^{n+1})}\big(1+O(q^{n+2})\big)\\
&=-(1\pm q^{n+1})(q^2;q^2)_\infty\big(1+O(q^{n+2})\big)\qquad\text{as}\quad n\to\infty,
\end{aligned}$$

whence

$$X_1I^{(n)}(1)+X_2I^{(n)}(-1)=-\big((X_1+X_2)+(X_1-X_2)q^{n+1}\big)(q^2;q^2)_\infty\big(1+O(q^{n+2})\big). \tag{25}$$

Therefore, the q-expansion of (25) starts either from q^0 if $X_1\ne X_2$, or from q^{n+1} otherwise. This means that the expression (25) is not zero for all n sufficiently large. $\square$

1.6. Everything is now ready for proving Theorem 1. Our general tool in deducing estimates for the linear independence measures will be Lemma 2.1 from [Ha], which needs the following 'q-adoption'.

Lemma 11. *Let γ_1,γ_2 be real numbers, and let*

$$\mathcal{I}_j^{(n)}=\mathcal{A}^{(n)}\gamma_j-\mathcal{B}_j^{(n)},\qquad j=1,2,\quad n=1,2,\dots,$$

be two sequences of linear forms with integer coefficients $\mathcal{A}^{(n)},\mathcal{B}_1^{(n)},\mathcal{B}_2^{(n)}$. Suppose that

$$\lim_{n\to\infty}\frac{\log|\mathcal{I}_1^{(n)}|}{n^2\log|p|}=\lim_{n\to\infty}\frac{\log|\mathcal{I}_2^{(n)}|}{n^2\log|p|}=-C_0,\qquad\lim_{n\to\infty}\frac{\log|\mathcal{A}^{(n)}|}{n^2\log|p|}=C_1$$

for positive numbers C_0, C_1, and that there exist infinitely many $n\in\mathbb{N}$ satisfying $\mathcal{I}_1^{(n)}/\mathcal{I}_2^{(n)}\ne\rho$ for any rational ρ. Then the numbers 1, γ_1 and γ_2 are linear independent over $\mathbb{Q}$ and, for any $\varepsilon>0$, there exists a positive integer $\mathcal{X}(\varepsilon)$ such that

$$|X_0+X_1\gamma_1+X_2\gamma_2|\geqslant X^{-C_1/C_0-\varepsilon},\qquad X=\max\{|X_1|,|X_2|\},$$

for any integers X_0,X_1,X_2 satisfying $X\geqslant\mathcal{X}(\varepsilon)$.

Proof of Theorem 1. Let $p\in\mathbb{Z}\setminus\{0,\pm1\}$. Take $x_1=1$, $x_2=-1$, and

$$\begin{aligned}
\mathcal{I}_j^{(n)}&=p^{-M}(p;p^2)_n\widehat{D}_{m,2n}(p)\cdot I^{(n)}(x_j)\\
&=\mathcal{A}^{(n)}L_1(x_j;q)-\mathcal{B}_j^{(n)},\qquad j=1,2,\quad n=1,2,\dots,
\end{aligned}$$

where $\mathcal{A}^{(n)}$, $\mathcal{B}_1^{(n)}$ and $\mathcal{B}_2^{(n)}$ are integers in accordance with Lemma 8. Then Lemmas 6, 7, 9 and the fact $\deg_p(p;p^2)_n = n^2$ imply

$$C_0 = -\lim_{n\to\infty} \frac{\log|\mathcal{I}_j^{(n)}|}{n^2\log|p|} = 3 - \frac{24}{\pi^2}, \qquad j = 1,2,$$
$$C_1 = \lim_{n\to\infty} \frac{\log|\mathcal{A}^{(n)}|}{n^2\log|p|} = 6 + \frac{24}{\pi^2}.$$

If ρ is a rational number, non-vanishing of $I_1^{(n)} - \rho I_2^{(n)}$ for all n sufficiently large follows from Lemma 10. Therefore, we can apply Lemma 11 to conclude with the linear independence of the numbers 1, $L_1(1;q)$, $L_1(-1;q)$, and with the estimate (1) for any integers X_0, X_1, X_2 satisfying $X = \max\{|X_1|, |X_2|\} \geqslant \mathcal{X}(\varepsilon)$. $\square$

2. Simultaneous approximations to the q-logarithm and q-dilogarithm

2.1. Now let $x_1 = x_2 = x$ in the settings of the previous section. Then all formulae, obtained there, remain valid, but we do not have any more simultaneous rational approximations to two q-logarithms, just to the one, $L_1(x;q)$. On the other hand, the function $\widetilde{R}(T)$ in (3) has now double zeros at the points $T = xq^t$ for $t = -1, -2, \dots, -n$, hence

$$\widetilde{R}'(T) = T\frac{\mathrm{d}}{\mathrm{d}T}\widetilde{R}(T) + (2n+1)\widetilde{R}(T) = \sum_{k=0}^{2n} \frac{A_k((2n+1) - 2nq^{k+n+1}T)}{(1-q^{k+n+1}T)^2} \tag{26}$$

has zeros at these points. Therefore, taking $R'(T) = \widetilde{R}'(T)\cdot T^{2n+1}$, we will consider the quantity

$$I'(x) = \sum_{t=0}^{\infty} R'(T)\big|_{T=xq^t} = \sum_{t=-n}^{\infty} R'(T)\big|_{T=xq^t}.$$

Lemma 12. *We have*

$$I'(x) = AL_2(x;q) - B^* - B^{**}, \tag{27}$$

where A is given in (10), *while*

$$B^* = \sum_{k=0}^{2n} A_k p^{(k+n+1)(2n+1)} \sum_{l=1}^{k} \frac{xp^l}{(p^l-x)^2}, \tag{28}$$
$$B^{**} = p^{(n+1)(2n+1)} \sum_{l=0}^{2n-1} \frac{(2n-l)x^{2n-l}}{p^{2n-l}-1} \sum_{k=0}^{2n} A_k q^{-k(l+1)}. \tag{29}$$

In other words, $I(x) = AL_1(x;q) - A^*(x) - A^{**}(x)$ and $I'(x)$ in (27) *viewed as functions of $p = 1/q$ realize simultaneous rational approximations to the q-logarithm $L_1(x;q)$ and q-dilogarithm $L_2(x;q)$.*

Proof. We obtain

$$\begin{aligned}
I'(x) &= \sum_{t=-n}^{\infty} R'(T)\big|_{T=xq^t} = \sum_{t=-n}^{\infty} x^{2n+1}q^{t(2n+1)} \sum_{k=0}^{2n} \frac{A_k((2n+1)-2nxq^{k+n+t+1})}{(1-xq^{k+n+t+1})^2} \\
&= \sum_{k=0}^{2n} A_k q^{-(k+n+1)(2n+1)} \sum_{t=-n}^{\infty} \frac{(2n+1)(xq^{k+n+t+1})^{2n+1} - 2n(xq^{k+n+t+1})^{2n+2}}{(1-xq^{k+n+t+1})^2} \\
&= \sum_{k=0}^{2n} A_k q^{-(k+n+1)(2n+1)} \sum_{l=k+1}^{\infty} \frac{(2n+1)(xq^l)^{2n+1} - 2n(xq^l)^{2n+2}}{(1-xq^l)^2}. \qquad (30)
\end{aligned}$$

Applying the identity

$$\frac{y-(2n+1)y^{2n+1}+2ny^{2n+2}}{(1-y)^2} = y\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{y-y^{2n+1}}{1-y}\right) = y\frac{\mathrm{d}}{\mathrm{d}y}\sum_{j=1}^{2n} y^j = \sum_{j=1}^{2n} jy^j, \quad |y|<1,$$

as in (12) we see that

$$\begin{aligned}
&\sum_{l=k+1}^{\infty} \frac{(2n+1)(xq^l)^{2n+1} - 2n(xq^l)^{2n+2}}{(1-xq^l)^2} \\
&\quad = \sum_{l=1}^{k+1} \frac{xq^l}{(1-xq^l)^2} - \sum_{l=k+1}^{\infty} \frac{xq^l - (2n+1)(xq^l)^{2n+1} + 2n(xq^l)^{2n+2}}{(1-xq^l)^2} \\
&\quad = \sum_{l=1}^{\infty} \frac{xq^l}{(1-xq^l)^2} - \sum_{l=1}^{k} \frac{xq^l}{(1-xq^l)^2} - \sum_{l=k+1}^{\infty}\sum_{j=1}^{2n} j(xq^l)^j \\
&\quad = L_2(x;q) - \sum_{l=1}^{k} \frac{xq^l}{(1-xq^l)^2} - \sum_{j=1}^{2n} \frac{jx^jq^{j(k+1)}}{1-q^j},
\end{aligned}$$

hence we may continue (30) as follows:

$$\begin{aligned}
I'(x) = \sum_{k=0}^{2n} A_k p^{(k+n+1)(2n+1)} \cdot L_2(x;q) - \sum_{k=0}^{2n} A_k p^{(k+n+1)(2n+1)} \sum_{l=1}^{k} \frac{xp^l}{(p^l-x)^2} \\
- p^{(n+1)(2n+1)} \sum_{j=1}^{2n} \frac{j(xq)^j}{1-q^j} \sum_{k=0}^{2n} A_k q^{-k(2n+1-j)}.
\end{aligned}$$

The coefficient of $L_2(x;q)$ in the latter expression is exactly the same as of $L_1(x;q)$ in (9), while for the tails we have the required formulae (28) and (29). $\square$

2.2. Using (7), (28) and the representation

$$\begin{aligned}
B^{**} = p^{(2n+1)^2} \sum_{l=0}^{2n-1} \frac{(2n-l)x^{2n-l}}{p^{2n-l}-1} \sum_{\substack{k_1=0 \\ k_1+k_2\leqslant l}}^{n} \sum_{k_2=0}^{n} \begin{bmatrix} 2n+l-k_1-k_2 \\ 2n \end{bmatrix}_p \begin{bmatrix} n \\ k_1 \end{bmatrix}_p \begin{bmatrix} n \\ k_2 \end{bmatrix}_p (-1)^{k_1+k_2} \\
\times x^{k_1+k_2-2n} p^{(n-k_1)(n-k_1+1)/2+(n-k_2)(n-k_2+1)/2}
\end{aligned}$$

derived from (29) and (17), we obtain the following assertion.

Lemma 13. *We have the inclusions*

$$X^{2n}(p;p)_{2n}p^{-(2n+1)^2}D_{2n}(p,x)^2\cdot B^*\in\mathbb{Z}[p], \tag{31}$$

$$X^{2n}p^{-(2n+1)^2}D_{2n}(p,1)\cdot B^{**}\in\mathbb{Z}[p], \tag{32}$$

where X is the numerator of the rational number x.

2.3. Reasoning as in the first paragraph of Subsection 1.5, we see that the inclusions of Lemmas 3, 5, 13 and the asymptotics of Lemma 6 give no chance to prove a linear independence result for the numbers 1, $L_1(x;q)$ and $L_2(x;q)$. Nevertheless, when $x=1$, inclusions (13) and (31) may be seriously improved, and this is a way to prove Theorem 2.

Lemma 14. *If $x=x_1=x_2=1$, then*

$$\begin{gathered}\begin{bmatrix}2n\\n\end{bmatrix}_p p^{-(2n+1)^2}\cdot A\in\mathbb{Z}[p], \qquad \begin{bmatrix}2n\\n\end{bmatrix}_p p^{-(2n+1)^2}D_{2n}(p,1)\cdot A^*\in\mathbb{Z}[p],\\ \begin{bmatrix}2n\\n\end{bmatrix}_p p^{-(2n+1)^2}D_{2n}(p,1)^2\cdot B^*\in\mathbb{Z}[p].\end{gathered} \tag{33}$$

In other words, the correct denominator of the approximations $I(1)$ and $I'(1)$ is

$$\begin{bmatrix}2n\\n\end{bmatrix}_p p^{-(2n+1)^2}D_{2n}(p,1)^2.$$

Proof. If $x=1$, then from (7)

$$\begin{aligned}\begin{bmatrix}2n\\n\end{bmatrix}_p\cdot A_kp^{(k+n+1)(2n+1)}&=p^{(2n+1)^2+k(k+1)/2}\frac{(p;p)_{2n}}{(p;p)_n^2}\cdot\frac{(p^{k+1};p)_n^2}{(p;p)_k(p;p)_{2n-k}}\\&=p^{(2n+1)^2+k(k+1)/2}\begin{bmatrix}n+k\\k\end{bmatrix}_p^2\begin{bmatrix}2n\\k\end{bmatrix}_p\in p^{(2n+1)^2}\mathbb{Z}[p].\end{aligned}$$

Substituting these formulae into (10), (28) and using (16), (32) result in (33). $\square$

2.4. It remains to indicate the results similar to Lemmas 6 and 10 for the sequences $I^{(n)}(1)=I(1)$ and $I'^{(n)}(1)=I'(1)$ when n increases to infinity.

Lemma 15. *For $p\in\mathbb{Z}\setminus\{0,\pm1\}$, we have*

$$\lim_{n\to\infty}\frac{\log|I^{(n)}(1)|}{n^2\log|p|}=\lim_{n\to\infty}\frac{\log|I'^{(n)}(1)|}{n^2\log|p|}=0.$$

In addition, for any pair of rational numbers X_1 and X_2, $X_1^2+X_2^2\ne0$, the condition

$$X_1I^{(n)}(1)+X_2I'^{(n)}(1)\ne0$$

holds unless n is sufficiently large.

Proof. As in the proof of Lemma 6, we have

$$I(1)=\widetilde{R}(1)+O(q^{2n+1})\quad\text{and}\quad I'(1)=\widetilde{R}'(1)+O(q^{2n+1})\qquad\text{as}\quad n\to\infty.$$

Formulae (26) give us

$$\begin{aligned}\widetilde{R}'(1)&=\widetilde{R}(1)\cdot\biggl((2n+1)-2\sum_{j=1}^{n}\frac{q^j}{1-q^j}+O(q^{n+1})\biggr)\\&=\widetilde{R}(1)\cdot\bigl((2n+1)+O(q)\bigr)\qquad\text{as}\quad n\to\infty,\end{aligned}$$

where the constant in $O(q)$ is independent of n. Therefore, the relation $X_1I^{(n)}(1)+X_2I'^{(n)}(1)=0$ cannot hold identically for a fixed pair of rationals X_1,X_2 with $X_1^2+X_2^2\ne0$, and also

$$\lim_{n\to\infty}\frac{\log|I^{(n)}(1)|}{n^2\log|q|}=\lim_{n\to\infty}\frac{\log|I'^{(n)}(1)|}{n^2\log|q|}=0,$$

thus showing the truth of the lemma. □

Proof of Theorem 2. Let $p\in\mathbb{Z}\setminus\{0,\pm1\}$ and $x=x_1=x_2=1$ in the above construction. Then the linear forms

$$\begin{aligned}\mathcal{I}_1^{(n)}&=\begin{bmatrix}2n\\n\end{bmatrix}_p p^{-(2n+1)^2}D_{2n}(p,1)^2\cdot I^{(n)}(1)=\mathcal{A}^{(n)}L_1(1;q)-\mathcal{B}_1^{(n)},\\ \mathcal{I}_2^{(n)}&=\begin{bmatrix}2n\\n\end{bmatrix}_p p^{-(2n+1)^2}D_{2n}(p,1)^2\cdot I'^{(n)}(1)=\mathcal{A}^{(n)}L_2(1;q)-\mathcal{B}_2^{(n)},\end{aligned}\qquad n=1,2,\dots,$$

have integer coefficients $\mathcal{A}^{(n)}$, $\mathcal{B}_1^{(n)}$ and $\mathcal{B}_2^{(n)}$ in accordance with Lemmas 8 and 14. The asymptotic behaviour of the linear forms and their coefficients is determined by Lemmas 15, 7 and the estimates

$$\deg_p\begin{bmatrix}2n\\n\end{bmatrix}_p=n^2,\quad \deg_p D_{2n}(p,1)=\frac{3}{\pi^2}(2n)^2+O(n\log n)\qquad\text{as}\quad n\to\infty$$

(for the latter one, see [BV1], Section 2, or [As]), hence

$$\begin{aligned}C_0&=-\lim_{n\to\infty}\frac{\log|\mathcal{I}_j^{(n)}|}{n^2\log|p|}=3-\frac{24}{\pi^2},\qquad j=1,2,\\ C_1&=\lim_{n\to\infty}\frac{\log|\mathcal{A}^{(n)}|}{n^2\log|p|}=6+\frac{24}{\pi^2}.\end{aligned}$$

If ρ is a rational number, non-vanishing of $I_1^{(n)}-\rho I_2^{(n)}$ for all n sufficiently large follows from the second part of Lemma 15. Therefore, we can apply Lemma 11 to conclude with the linear independence of the numbers 1, $\zeta_q(1)=L_1(1;q)$, $\zeta_q(2)=L_2(1;q)$, and with the estimate (2) for any integers X_0,X_1,X_2 satisfying $X=\max\{|X_1|,|X_2|\}\geqslant\mathcal{X}(\varepsilon)$. □

References

[As] W. Van Assche, *Little q-Legendre polynomials and irrationality of certain Lambert series*, Ramanujan J. **5** (2001), no. 3, 295–310.

[Be] J.-P. Bézivin, *Indépendence linéaire des valeurs des solutions transcendantes de certaines équations fonctionelles*, Manuscripta Math. **61** (1988), 103–129.

[Bo] P. Borwein, *On the irrationality of* $\sum(1/(q^n+r))$, J. Number Theory **37** (1991), 253–259.

[BV1] P. Bundschuh and K. Väänänen, *Arithmetical investigations of a certain infinite product*, Compositio Math. **91** (1994), 175–199.

[BV2] P. Bundschuh and K. Väänänen, *On the linear independence of q-analogues of certain classical constants*, Manuscript (2004).

[BZ] P. Bundschuh and W. Zudilin, *Rational approximations to a q-analogue of π and some other q-series*, Preprint (2004), submitted.

[Er] P. Erdős, *On arithmetical properties of Lambert series*, J. Indiana Math. Soc. **12** (1948), 63–66.

[GR] G. Gasper and M. Rahman, *Basic hypergeometric series*, Encyclopedia of Mathematics and its Applications, vol. 35, Cambridge Univ. Press, Cambridge, 1990.

[Ha] M. Hata, *Rational approximations to π and some other numbers*, Acta Arith. **63** (1993), no. 4, 335–349.

[MVZ] T. Matalo-aho, K. Väänänen and W. Zudilin, *New irrationality measures for q-logarithms*, Math. Comput. (2004), submitted.

[PA] K. Postelmans and W. Van Assche, *Irrationality of $\zeta_q(1)$ and $\zeta_q(2)$*, Manuscript (2004).

[Zu1] W. Zudilin, *On the irrationality measure for a q-analogue of $\zeta(2)$*, Mat. Sb. [Russian Acad. Sci. Sb. Math.] **193** (2002), no. 8, 49–70.

[Zu2] W. Zudilin, *Diophantine problems for q-zeta values*, Mat. Zametki [Math. Notes] **72** (2002), no. 6, 936–940.

[Zu3] W. Zudilin, *Heine's basic transform and a permutation group for q-harmonic series*, Acta Arith. **111** (2004), no. 2, 153–164.

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