HYPERGEOMETRIC EQUATION AND RAMANUJAN FUNCTIONS

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Abstract. In this paper we give analogues of the Ramanujan functions and nonlinear differential equations for them. Investigating a modular structure of solutions for nonlinear differential systems, we deduce new identities between the Ramanujan and hypergeometric functions. Another result of this paper is a solution of transcendence problems concerning nonlinear systems.

In 1916 S. Ramanujan has proved [12] that the functions

$$P(q) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n,$$
$$Q(q) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n, \qquad R(q) = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n, \tag{1}$$

where $\sigma_k(n) = \sum_{d|n} d^k$, satisfy the system of nonlinear differential equations

$$q\frac{\mathrm{d}P}{\mathrm{d}q} = \frac{1}{12}(P^2 - Q), \qquad q\frac{\mathrm{d}Q}{\mathrm{d}q} = \frac{1}{3}(PQ - R), \qquad q\frac{\mathrm{d}R}{\mathrm{d}q} = \frac{1}{2}(PR - Q^2) \tag{2}$$

(see also [5, Chapter X, Sect. 5]). Note that Q and R are modular as functions of $\tau = \frac{1}{2\pi i}\log q$.

The hypergeometric function

$$F(\alpha,\beta;\gamma;z) = \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\cdot\frac{z^n}{n!},$$

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where $(\alpha)_0 = 1$ and $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ for $n = 1, 2, \dots$, satisfies the second order linear homogeneous differential equation

$$z(1-z)\frac{\mathrm{d}^2V}{\mathrm{d}z^2} + (\gamma - (\alpha+\beta+1)z)\frac{\mathrm{d}V}{\mathrm{d}z} - \alpha\beta V = 0 \tag{3}$$

(see, e.g., [15, Chapter 14, Sect. 14.2]). A method starting from original works of K. Jacobi, H. A. Schwarz, M. Halphen et al. allows to derive relations between solutions of the hypergeometric equation (3) and of Ramanujan-like nonlinear systems. To illustrate this statement, we give the formula

$$F\biggl(\frac{1}{12}, \frac{5}{12}; 1; \frac{Q^3-R^2}{Q^3}\biggr) = Q^{1/4} \tag{4}$$

(see [14] or the end of Sect. 4 below).

This paper deals with both nonlinear systems satisfied by modular functions, and identities between these functions and hypergeometric functions for a special choice of parameters. The recent papers [6, 11, 4] are devoted to the study of similar questions, but we have another aim. Our work is mostly inspired by Yu. V. Nesterenko's result [9] in number theory about algebraic independence over $\mathbb{Q}$ of at least three numbers among q, $P(q)$, $Q(q)$, and $R(q)$ for any $q \in \mathbb{C}$, $0 < |q| < 1$. Nesterenko's proof is based on the following three crucial features for the Ramanujan functions:

(i) algebraic independence over $\mathbb{C}(q)$;

(ii) algebraic system of differential equations (2) satisfied by the collection of functions in question;

(iii) *polynomial* growth of *integral* coefficients of the Taylor expansions with respect to q.

There exist a lot of another examples of functions satisfying conditions (i)–(iii); some of them are indicated below. However, any such example has the modular nature, that is, the corresponding functions are algebraic over the field generated by the Ramanujan functions. The question about the existence of "nonmodular" functions of one variable q satisfying (i)–(iii) is still unanswered.

1. Hypergeometric equation and nonlinear systems

Equation (3) has three regular singular points at $0, 1$, and ∞. Making the change

$U = z^{\gamma/2}(1-z)^{(\alpha+\beta+1-\gamma)/2}V$ in (3) we deduce the equation

$$U'' + \left(\frac{a+c}{4z^2} + \frac{b+c}{4(z-1)^2} - \frac{c}{2z(z-1)}\right)U = 0, \tag{5}$$

where

$$a = \gamma(1-\alpha-\beta) + 2\alpha\beta, \qquad b = (\alpha+\beta)(\gamma-\alpha-\beta) + 2\alpha\beta - \gamma + 1,$$
$$c = \gamma(\alpha+\beta+1-\gamma) - 2\alpha\beta.$$

Let $\pi i\tau$ be the ratio of two independent solutions

$$u_1 = u_1(z) \qquad \text{and} \qquad u_0 = z^{\gamma/2}(1-z)^{(\alpha+\beta+1-\gamma)/2}F(\alpha,\beta;\gamma;z) \tag{6}$$

of (5), and $\delta = \frac{1}{\pi i}\mathrm{d}/\mathrm{d}\tau$. Then the functions

$$y_0 = \delta\log u_0, \qquad y_1 = \delta\log\frac{u_0}{z}, \qquad \text{and} \qquad y_2 = \delta\log\frac{u_0}{z-1} \tag{7}$$

satisfy the system of nonlinear differential equations

$$\begin{aligned}
\delta Y_0 &= Y_0^2 - \frac{a}{4}(Y_0-Y_1)^2 - \frac{b}{4}(Y_0-Y_2)^2 - \frac{c}{4}(Y_1-Y_2)^2,\\
\delta Y_1 &= Y_1^2 - \frac{a}{4}(Y_0-Y_1)^2 - \frac{b}{4}(Y_0-Y_2)^2 - \frac{c}{4}(Y_1-Y_2)^2,\\
\delta Y_2 &= Y_2^2 - \frac{a}{4}(Y_0-Y_1)^2 - \frac{b}{4}(Y_0-Y_2)^2 - \frac{c}{4}(Y_1-Y_2)^2
\end{aligned} \tag{8}$$

(see [11, Sect. 3, Example 2]).

In the case of hypergeometric equation with parameters $\alpha = \beta = \frac{1}{2}$ and $\gamma = 1$ we obtain $a = b = c = \frac{1}{2}$, and after the substitution

$$\Psi_2 = \frac{1}{4}(Y_0+Y_2), \qquad \Psi_3 = \frac{1}{4}(Y_1+Y_2), \qquad \Psi_4 = \frac{1}{4}(Y_0+Y_1)$$

in (8) we get the Halphen system [3]

$$\delta(\Psi_2+\Psi_3) = 4\Psi_2\Psi_3, \qquad \delta(\Psi_2+\Psi_4) = 4\Psi_2\Psi_4, \qquad \delta(\Psi_3+\Psi_4) = 4\Psi_3\Psi_4. \tag{9}$$

An easy computation using classical formulas (see [15, Sect. 21.8] and [11, Sect. 3, Example 4]) shows that the system (9) is satisfied by the logarithmic δ-derivatives of theta constants

$$\psi_2 = \frac{\delta\vartheta_2}{\vartheta_2}, \qquad \psi_3 = \frac{\delta\vartheta_3}{\vartheta_3}, \qquad \psi_4 = \frac{\delta\vartheta_4}{\vartheta_4} \tag{10}$$

(for other proofs of this fact we refer to [3] and [16]).

2. Gauss–Schwarz theory

Let each of the numbers

$$\lambda=|1-\gamma|=\sqrt{1-a-c}, \qquad \mu=|\alpha+\beta-\gamma|=\sqrt{1-b-c},$$
$$\nu=|\alpha-\beta|=\sqrt{1-a-b}$$

be either zero or the reciprocal of an integer greater than 1, and $\lambda+\mu+\nu<1$. Then the projective monodromy group Γ of (3) is isomorphic to a certain finitely generated subgroup of $SL_2(\mathbb{R})$ (the so-called *Schwarz triangle group*), and the map $\tau\mapsto z(\tau)$ is an *automorphic function with respect to* Γ, well defined inside the Γ-stable circle Ω:

$$z(\tau)=z(\gamma\tau)=z\biggl(\frac{a\tau+b}{c\tau+d}\biggr), \qquad \gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma, \quad \tau\in\Omega.$$

In particular, the function $z(\tau)$ is meromorphic in Ω.

The Wronskian of the equation (5) is a constant C; therefore,

$$\frac{\mathrm{d}\tau}{\mathrm{d}z}=\frac{u_1'u_0-u_1u_0'}{\pi i u_0^2}=\frac{C}{\pi i u_0^2}. \tag{11}$$

Hence $u_0^2=\frac{C}{\pi i}\mathrm{d}z/\mathrm{d}\tau$, which yields

$$u_0^2\bigl(z(\tau)\bigr)=C\cdot\delta z(\tau), \qquad \tau\in\Omega. \tag{12}$$

In the right-hand side of (12) we see an automorphic function of weight 2 with respect to Γ (see, e.g., [2, Sect. 44]). The following statement shows how to express the identity (12) in terms of the solution (7) of the system (8).

Lemma 1. *Let y_0,y_1,y_2 be a solution of* (8) *assigned to the pair of linearly independent solutions* (6) *of* (5). *Then*

$$u_0^2\biggl(\frac{y_2(\tau)-y_0(\tau)}{y_2(\tau)-y_1(\tau)}\biggr)=\frac{(y_0(\tau)-y_1(\tau))(y_2(\tau)-y_0(\tau))}{y_2(\tau)-y_1(\tau)}, \qquad \tau\in\Omega. \tag{13}$$

Proof. By (11) and (7) we get

$$y_0-y_1=\delta\log z=\frac{u_0^2}{Cz}, \qquad y_0-y_2=\delta\log(z-1)=\frac{u_0^2}{C(z-1)},$$

which implies $z=(y_2-y_0)/(y_2-y_1)$. Therefore,

$$u_0^2\biggl(\frac{y_2-y_0}{y_2-y_1}\biggr)=u_0^2(z)=Cz(y_0-y_1)=\frac{(y_0-y_1)(y_2-y_0)}{y_2-y_1},$$

as required in (13).

Proposition 1. *The following equality holds*:

$$F^2\left(\alpha,\beta;\gamma;\frac{y_2-y_0}{y_2-y_1}\right)=(y_0-y_1)^{\gamma-\alpha-\beta}(y_2-y_1)^{\alpha+\beta}(y_2-y_0)^{1-\gamma}. \tag{14}$$

Proof. By Lemma 1 and the formula

$$\begin{aligned}F^2\left(\alpha,\beta;\gamma;\frac{y_2-y_0}{y_2-y_1}\right)&=z^{-\gamma}(1-z)^{-\alpha-\beta-1+\gamma}u_0^2(z)\\&=\left(\frac{y_2-y_0}{y_2-y_1}\right)^{-\gamma}\left(\frac{y_0-y_1}{y_2-y_1}\right)^{-\alpha-\beta-1+\gamma}u_0^2\left(\frac{y_2-y_0}{y_2-y_1}\right)\end{aligned} \tag{15}$$

we obtain the desired relation (14).

Remark. Formula (15) explains what branches of the root functions one must consider in (14). Their choice is determined by our choice of branches of $z^{-\gamma}$ and $(1-z)^{-\alpha-\beta-1+\gamma}$, which we assume to take real values for $z\in(0,1)$. It is interesting to note that some corollaries of (14) (for instance, (4)) can sometimes be regarded as equalities of formal power series, without mention of branches of root functions.

Proposition 2. *The functions* (7) *are meromorphic in* Ω; *moreover*, *they have only simple poles. If* $\tau_0\in\Omega$ *is a pole of any of the functions* (7) *then*

$$z(\tau_0)\in\{0,1,\infty\}. \tag{16}$$

Proof. According to formulas (7) and (12) we have

$$y_0(\tau)=\frac{1}{2}\delta\log\bigl(\delta z(\tau)\bigr),\qquad y_1(\tau)=\frac{1}{2}\delta\log\frac{\delta z(\tau)}{z(\tau)},\qquad y_2(\tau)=\frac{1}{2}\delta\log\frac{\delta z(\tau)}{z(\tau)-1}. \tag{17}$$

Hence *simple* poles of functions (17) coincide with zeros and poles of $\delta z(\tau)$, $z(\tau)$, and $z(\tau)-1$. The poles of the last functions are poles of $z(\tau)$, which satisfy (16) (that is, they belong to the set of cusps of Γ). Zeros of a function $z(\tau)-z_0$, where $z_0\in\{0,1\}$, are exactly points τ_0 such that $z(\tau_0)=z_0$. Finally, it remains to prove that all zeros of the function (12) satisfy (16). Assuming the contrary, suppose that there exists τ_0 such that both $u_0(z(\tau_0))=0$ and $z_0=z(\tau_0)\notin\{0,1,\infty\}$. Since z_0 is not a singular point of the differential equation (5), linearly independent solutions

$u_0(z)$ and $u_1(z)$ are holomorphic at $z = z_0$, and $u_1(z_0) \neq 0$ because $u_0(z_0) = 0$. Therefore the point $\tau_0 = \tau(z_0) = \frac{1}{\pi i} u_1(z_0)/u_0(z_0) = \infty$ does not belong to Ω. This contradiction completes the proof.

In the proof of Lemma 1 we express the automorphic function $z(\tau)$ in terms of the solution of the nonlinear system (8). The functions (7) are not automorphic, but they have transformations under the action of Γ as follows.

Proposition 3. *Suppose that* Γ *is the monodromy group for* (5) *and the collection* $y_0(\tau), y_1(\tau), y_2(\tau)$ *gives a solution of the dual system* (8)*. Then*

$$\begin{gathered} y_j(\gamma\tau) = (c\tau + d)^2 y_j(\tau) + \frac{1}{\pi i} c(c\tau + d), \qquad j = 0, 1, 2, \\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \quad \tau \in \Omega. \end{gathered} \tag{18}$$

Proof. Without loss of generality we give the proof of (18) only for the case of $y_0(\tau)$. The action of the monodromy group on a pair of linearly independent solutions $u_0(z)$ and $u_1(z)$ can be expressed as

$$\gamma\colon u_0(z) \mapsto c u_1(z) + d u_0(z), \quad \gamma\colon u_1(z) \mapsto a u_1(z) + b u_0(z), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Consequently, for $\gamma \in \Gamma$ using (7) we obtain

$$\begin{aligned} y_0(\gamma\tau) &= \frac{1}{\pi i} \frac{\mathrm{d}}{\mathrm{d}(\gamma\tau)} \log u_0\big(z(\gamma\tau)\big) = \left(\frac{\mathrm{d}(\gamma\tau)}{\mathrm{d}\tau}\right)^{-1} \cdot \frac{1}{\pi i} \frac{\mathrm{d}}{\mathrm{d}\tau} \log(c u_1 + d u_0) \\ &= (c\tau + d)^2 \cdot \frac{1}{\pi i} \frac{\mathrm{d}}{\mathrm{d}\tau} \log\big(u_0 \cdot (c\tau + d)\big) \\ &= (c\tau + d)^2 \cdot \left(\delta \log u_0 + \frac{c}{\pi i (c\tau + d)}\right), \qquad \tau \in \Omega, \end{aligned}$$

which is the desired conclusion.

By (18) and Proposition 2, the differences $y_0 - y_1$ and $y_0 - y_2$ are *automorphic forms of weight* 2 *with respect to* Γ:

$$\begin{gathered} (y_0 - y_j)(\gamma\tau) = (c\tau + d)^2 \cdot (y_0 - y_j)(\tau), \qquad j = 1, 2, \\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \quad \tau \in \Omega. \end{gathered}$$

Thus, the identity (13) shows what automorphic function (of weight 0) one must substitute for z in $u_0^2(z)$ to get an automorphic function of weight 2.

Another advantage of the functions $y_0(\tau), y_1(\tau), y_2(\tau)$ is the possibility to construct new automorphic functions with their help.

Proposition 4. *Suppose that Ω is the interior of the circle stable under the Schwarz triangle group Γ, and a function $f(\tau)$ meromorphic in Ω satisfies the functional equation*

$$f(\gamma\tau) = (c\tau + d)^2 f(\tau) + \frac{k}{\pi i} c(c\tau + d), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \quad \tau \in \Omega,$$

where $k > 0$. Then

(a) *the differential operator*

$$\delta_p = \delta - \frac{p}{k} f(\tau)$$

takes each automorphic function of weight p with respect to Γ to an automorphic function of weight $p + 2$;

(b) *the function $\delta_1 f(\tau)$ is a automorphic form of weight 4 with respect to Γ.*

We omit the proof since there is a detailed description in [5, Chapter X, Sect. 5] for the concrete function $f(\tau) = P(e^{2\pi i\tau})$ satisfying the functional equation

$$f(\gamma\tau) = (c\tau + d)^2 f(\tau) + \frac{6}{\pi i} c(c\tau + d), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \quad \Im\tau > 0.$$

We observe that if $f(\tau)$ satisfies the assumptions of Proposition 4 and is a *holomorphic* function in Ω, then, by this result, one obtains a sequence of automorphic forms $f_4 = \delta_1 f$, $f_6 = \delta_4 f_4$, $f_8 = \delta_6 f_6$, etc. of weights $4, 6, 8, \ldots$ respectively. This fact (and the known structure of modular forms for $SL_2(\mathbb{Z})$) gives one arguments to prove the system of differential equations (2).

3. Isomorphism of differentially stable fields

In this section we apply the one-dimensional case of the general method introduced in the joint work [1] of D. Bertrand and this author to compute the transcendence degree of the differential field generated by automorphic forms. Reducing to the case of dimension 1 enables us to be more explicit.

We fix a hypergeometric equation (3) such that its projective monodromy group Γ is the Schwarz triangle group with angles λ, μ, and ν. Let Ω be the interior of

Γ-stable circle. We take the nonlinear system (8) dual to (3) (or, equivalently, to (5)).

We take an *arbitrary* solution $y_0(\tau), y_1(\tau), y_2(\tau)$ of (8) that is analytic in a neighborhood of $\tau_0 \in \Omega$ and the quantities $y_0(\tau_0), y_1(\tau_0), y_2(\tau_0)$ are all distinct. Consider the field

$$\mathcal{L} = \mathbb{C}\big(\tau, y_0(\tau), y_1(\tau), y_2(\tau)\big),$$

which is δ-differentially stable by (8). Note that the rings $\mathbb{C}[\tau, y_0, y_1, y_2]$ and $\mathbb{C}[y_0, y_1, y_2]$ are also δ-stable.

By [11, Theorem 1.1, 3)] we can find linearly independent solutions u_0, u_1 of the homogeneous equation (5) such that setting $\pi i\tau = u_1/u_0$ we obtain by (7) the functions $y_0(\tau), y_1(\tau), y_2(\tau)$ (at least in the neighborhood of τ_0). Consider the $\mathrm{d}/\mathrm{d}z$-differentially stable field

$$\mathcal{K} = \mathbb{C}\big(z, u_0(z), u_0'(z), u_1(z), u_1'(z)\big)$$

which is the Picard–Vessiot extension for the equation (5).

The fields $\mathcal{K}$ and $\mathcal{L}$ are defined *globally*: for $\mathbb{C} \setminus \{0, 1\}$ and for Ω, respectively (the last fact is a consequence of Gauss–Schwarz theory).

Lemma 2. *There holds the embedding $\mathcal{L} \subset \mathcal{K}$.*

Proof. By (11) and (7) one has

$$y_0 = \frac{u_0' u_0}{C}, \qquad y_1 = \frac{u_0' u_0}{C} - \frac{u_0^2}{Cz}, \qquad y_2 = \frac{u_0' u_0}{C} - \frac{u_0^2}{C(z-1)}, \tag{19}$$

which yields that $\tau = \frac{1}{\pi i} u_1/u_0$ and the functions $y_0(\tau)$, $y_1(\tau)$, and $y_2(\tau)$ belong to $\mathbb{C}(z, u_0, u_0', u_1) = \mathcal{K}$. This proves the required embedding.

In general, the inverse embedding $\mathcal{K} \subset \mathcal{L}$ fails, but this drawback can be easily remedied.

Consider the *symmetric square* or the *Appell transform* (see [15, Chapter 14, Example 10]) of the differential equation (5), that is the third order linear homogeneous differential equation with fundamental system of solutions u_0^2, u_0u_1, u_1^2. Its Picard–Vessiot extension is precisely

$$\mathcal{K}_2 = \mathbb{C}\big(z, u_0^2, (u_0^2)', (u_0^2)'', u_1^2, (u_1^2)', (u_1^2)'', u_0u_1, (u_0u_1)', (u_0u_1)''\big).$$

Proposition 5. *The differentially stable fields $\mathcal{K}_2$ and $\mathcal{L}$ coincide.*

Proof. Using formulas (19) and $\tau = u_0u_1/u_0^2$, we deduce that the functions τ, $y_0(\tau)$, $y_1(\tau)$, and $y_2(\tau)$ belong to $\mathbb{C}(z, u_0^2, (u_0^2)', u_0u_1) \subset \mathcal{K}_2$. This yields the embedding $\mathcal{L} \subset \mathcal{K}_2$.

Conversely, by (19) we get $z = (y_2 - y_0)/(y_2 - y_1)$ (see the proof of Lemma 1) and

$$u_0^2 = Cz(y_0 - y_1), \qquad u_1^2 = \tau^2 u_0^2, \qquad u_0u_1 = \tau u_0^2;$$

the $\mathrm{d}/\mathrm{d}z$-stability of $\mathcal{L}$ follows from the formula

$$\frac{\mathrm{d}}{\mathrm{d}z} = \frac{\mathrm{d}\tau}{\mathrm{d}z}\frac{\mathrm{d}}{\mathrm{d}\tau} = \left(\frac{u_1}{u_0}\right)'\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{C}{u_0^2}\delta$$

and the δ-stability of $\mathcal{L}$. This implies the embedding $\mathcal{K}_2 \subset \mathcal{L}$ and completes the proof of the proposition.

Corollary 1. *The field $\mathcal{L}$ does not depend on the choice of the solution $y_0(\tau), y_1(\tau)$, $y_2(\tau)$ of the system* (8) *analytic in a neighborhood of $\tau_0 \in \Omega$ with distinct quantities $y_0(\tau_0), y_1(\tau_0), y_2(\tau_0)$.*

Proof. Since Picard–Vessiot extensions do not depend on the choice of linearly independent solutions u_0, u_1 of (5), the application of Proposition 5 completes the proof.

Corollary 2. *The transcendence degree of $\mathcal{L}$ over $\mathbb{C}$ is* 4*.*

Proof. First we have

$$\operatorname{tr\,deg}_{\mathbb{C}} \mathcal{L} = \operatorname{tr\,deg}_{\mathbb{C}} \mathcal{K}_2 = \operatorname{tr\,deg}_{\mathbb{C}} \mathcal{K} = \operatorname{tr\,deg}_{\mathbb{C}(z)} \mathcal{K} + 1.$$

The transcendence degree of the Picard–Vessiot extension $\mathcal{K}$ over $\mathbb{C}(z)$ is equal to the dimension of the Galois group for the differential equation (5), which is the dimension of the monodromy group Γ. Since the dimension of any Schwarz triangle group is 3, we get the desired conclusion.

The result of Corollary 2 (in slightly distinct notation) is classical and has many different proofs (see [7, 10] for a more general assertion). The proofs in this section demonstrate one-dimensional potentials of the method from [1], which seems to be new even in this simple case.

4. Analogues of Ramanujan systems

We are interested in finding hypergeometric equations (3) such that the dual system (8) possesses solutions having expansions in non-negative powers of $q = e^{\pi i\tau}$ (where $\Im\tau > 0$) with almost all integral coefficients. We can restrict considerations to the case $\gamma = 1$ (equivalently, $a + c = 1$) since all other cases correspond to permutations of the parameters a, b, c (see Sect. 2 above). Our numerical results for systems (8) with periodic and good-arithmetic solutions (the total number of these systems is 9 by [4]) are gathered in the table below.

The cases considered above of logarithmic δ-derivatives of theta constants (10) ($\lambda = \mu = \nu = 0$) and the Ramanujan functions (1) ($\lambda = 0$, $\mu = \frac{1}{2}$, $\nu = \frac{1}{3}$) are related by the formulas

$$\begin{aligned} P &= 4(\psi_2 + \psi_3 + 4\psi_4), \\ Q &= 4^2\big((\psi_2 + 7\psi_3 - 8\psi_4)^2 - 48(\psi_3 - \psi_4)^2\big) = (\vartheta_2^4 + 8\vartheta_4^4)^2 - 48\vartheta_2^8, \\ R &= 4^3(\psi_2 + \psi_3 - 2\psi_4)\big((\psi_2 - 17\psi_3 + 16\psi_4)^2 - 288(\psi_3 - \psi_4)^2\big) \\ &= (\vartheta_2^4 + \vartheta_3^4)\big((\vartheta_4^4 - 16\vartheta_2^4)^2 - 288\vartheta_2^8\big). \end{aligned} \tag{20}$$

We have not succeeded in the search of references for (20) whereas the relations between $P(q^2), Q(q^2), R(q^2)$ and $\psi_2(q), \psi_3(q), \psi_4(q)$ are well known (see, e.g., [16]). One possible way to prove identities (20) is to use the "modularity" of (1) and (10) and to check the equality of their first terms in q-expansions; another way is based on the differential equations for the functions (1) and (10).

It is clear that the functions

$$\begin{gathered} P_2(q) = \frac{P(q) + 2P(q^2)}{1+2}, \quad Q_2(q) = \frac{Q(q) + 2^2Q(q^2)}{1+2^2}, \\ R_2(q) = \frac{R(q) + 2^3R(q^2)}{1+2^3}, \quad S_2(q) = \frac{R_2^2(q)}{Q_2(q)}, \end{gathered} \tag{21}$$

$$\begin{gathered} P_3(q) = \frac{P(q) + 3P(q^3)}{1+3}, \quad Q_3(q) = \frac{Q(q) + 3^2Q(q^3)}{1+3^2}, \quad R_3(q) = \frac{R(q) + 3^3R(q^3)}{1+3^3}, \\ S_3(q) = \frac{R_3^2(q)}{Q_3(q)}, \quad T_3(q) = \frac{R_3^3(q)}{Q_3^2(q)}, \quad U_3(q) = \frac{R_3^4(q)}{Q_3^3(q)} \end{gathered} \tag{22}$$

have Taylor expansions with respect to q with integral coefficients of polynomial growth (cf. the assumption (iii) from the introductory section). To check the assumptions (i) and (ii) we give the following assertions.

Proposition 6. *The functions* P_2, Q_2, R_2, S_2 *satisfy the differential equations*

$$\delta P_2 = \frac{1}{8}(P_2^2 - Q_2), \quad \delta Q_2 = \frac{1}{2}(P_2 Q_2 - R_2),$$
$$\delta R_2 = \frac{1}{4}(3P_2R_2 - 2Q_2^2 - S_2), \quad \delta S_2 = P_2 S_2 - Q_2 R_2$$

and the additional condition $R_2^2 = Q_2 S_2$. *The functions* P_2, Q_2, R_2 *are algebraically independent over* $\mathbb{C}(q)$.

Proposition 7. *The functions* $P_3, Q_3, R_3, S_3, T_3, U_3$ *satisfy the differential equations*

$$\delta P_3 = \frac{1}{6}(P_3^2 - Q_3), \quad \delta Q_3 = \frac{2}{3}(P_3 Q_3 - R_3),$$
$$\delta R_3 = \frac{1}{2}(2P_3R_3 - Q_3^2 - S_3), \quad \delta S_3 = \frac{1}{3}(4P_3S_3 - 3Q_3R_3 - T_3),$$
$$\delta T_3 = \frac{1}{6}(10P_3T_3 - 9R_3^2 - U_3), \quad \delta U_3 = 2(P_3U_3 - R_3S_3) = 2(P_3U_3 - Q_3T_3)$$

and the additional conditions $R_3^2 = Q_3S_3$, $R_3S_3 = Q_3T_3$, $S_3^2 = Q_3U_3$. *The functions* P_3, Q_3, R_3 *are algebraically independent over* $\mathbb{C}(q)$.

Propositions 6 and 7 can be derived directly either from the system of differential equations (2), or from the systems (8) for Schwarz triangle groups with angles $\lambda = 0$, $\mu = \frac{1}{2}$, $\nu = \frac{1}{4}$ and $\lambda = 0$, $\mu = \frac{1}{2}$, $\nu = \frac{1}{6}$ and the table.

We now want to explain the origin of relations from the last column of the table. Numerical computations using an easy algorithm written for GP-PARI calculator gives one q-expansions of the functions $y_0 = \frac{1}{2} - c \cdot Cq + O(q^2)$, $y_1 = -\frac{1}{2} + O(q^2)$, and $y_2 = \frac{1}{2} + a \cdot Cq + O(q^2)$ up to q^n for arbitrary $n \geq 1$; here $C > 0$ is some specially selected constant. By Proposition 2, the functions y_0, y_1, y_2 have at most one (if $\lambda = \mu = 0$) or two (if $\lambda = 0$) simple poles in the upper half-plane $\Omega = \{\tau : \Im\tau > 0\}$; therefore there exist two (if $\lambda = \mu = 0$) or one (if $\lambda = 0$) linear combinations of y_0, y_1, y_2 representing functions *holomorphic* in Ω. To identify them with P, P_2, or P_3 it is sufficient to look only at finitely many, say seven, terms in their q-expansions. Indeed, the linear spaces $S_k(\Gamma)$, where $k = 2$, 4, or 6, of cusp forms (which, in particular, contain automorphic forms of weight k vanishing at $q = 0$) have dimensions ≤ 6 by [13, Theorem 2.24] in all cases listed in the table. Thus, any difference of the functions separated by sign '=' in the last column of the table

TABLE. Solutions of nonlinear systems (8) related to Ramanujan functions

λ,μ,ν	$\alpha,\beta;\gamma$	a,b,c	Ramanujan systems
$0,0,0$	$\frac12,\frac12;1$	$\frac12,\frac12,\frac12$	$5y_0+5y_1+2y_2=P$ $3y_0+3y_1+2y_2=P_2$
$0,0,\frac12$	$\frac14,\frac34;1$	$\frac38,\frac38,\frac58$	$4y_0+5y_1+3y_2=P$ $2y_0+3y_1+3y_2=P_2$
$0,0,\frac13$	$\frac13,\frac23;1$	$\frac49,\frac49,\frac59$	$4y_0+5y_1+3y_2=P$ $2y_0+2y_1+2y_2=P_3$ $(y_2-y_1)^2=Q_3$ $(y_2-y_1)^2(2y_0-y_1-y_2)=R_3$
$0,\frac12,\frac13$	$\frac1{12},\frac5{12};1$	$\frac{41}{72},\frac{23}{72},\frac{31}{72}$	$4y_0+5y_1+3y_2=P$ $(y_0-y_1)(y_2-y_1)=Q$ $(y_0-y_1)^2(y_2-y_1)=R$
$0,\frac12,\frac14$	$\frac18,\frac38;1$	$\frac{19}{32},\frac{11}{32},\frac{13}{32}$	$3y_0+3y_1+2y_2=P_2$ $(y_0-y_1)(y_2-y_1)=Q_2$ $(y_0-y_1)^2(y_2-y_1)=R_2$
$0,\frac12,\frac16$	$\frac16,\frac13;1$	$\frac{11}{18},\frac{13}{36},\frac7{18}$	$\frac52y_0+2y_1+\frac32y_2=P_3$ $(y_0-y_1)(y_2-y_1)=Q_3$ $(y_0-y_1)^2(y_2-y_1)=R_3$
$0,\frac13,\frac13$	$\frac16,\frac12;1$	$\frac12,\frac7{18},\frac12$	$2y_0+2y_1+2y_2=P(-q^2)$ $(y_0-y_1)(y_2-y_1)=Q(-q^2)$ $\bigl(\frac12(y_0-y_1)+\frac12(y_2-y_1)\bigr)$ $\times(y_0-y_1)(y_2-y_1)=R(-q^2)$
$0,\frac14,\frac14$	$\frac14,\frac12;1$	$\frac12,\frac7{16},\frac12$	$\frac32y_0+y_1+\frac32y_2=P_2(-q^2)$ $(y_0-y_1)(y_2-y_1)=Q_2(-q^2)$ $\bigl(\frac12(y_0-y_1)+\frac12(y_2-y_1)\bigr)$ $\times(y_0-y_1)(y_2-y_1)=R_2(-q^2)$
$0,\frac16,\frac16$	$\frac13,\frac12;1$	$\frac12,\frac{17}{36},\frac12$	$\frac54y_0+\frac12y_1+\frac54y_2=P_3(-q^2)$ $(y_0-y_1)(y_2-y_1)=Q_3(-q^2)$ $\bigl(\frac12(y_0-y_1)+\frac12(y_2-y_1)\bigr)$ $\times(y_0-y_1)(y_2-y_1)=R_3(-q^2)$

is an automorphic form, which is essentially zero since it has order $O(q^7)$ and lies in the corresponding space $S_k(\Gamma)$.

Finally, we obtain Q- and R-functions by Propositions 3 and 4. This completes the substantiation of the table.

In addition to this table we present some identities produced by Proposition 1:

$$F\left(\frac{1}{3},\frac{2}{3};1;\frac{Q_3^{3/2}-R_3}{2Q_3^{3/2}}\right)=Q_3^{1/4},\qquad F\left(\frac{1}{8},\frac{3}{8};1;\frac{Q_2^3-R_2^2}{Q_2^3}\right)=Q_2^{1/4},\qquad (23)$$
$$F\left(\frac{1}{6},\frac{1}{3};1;\frac{Q_3^3-R_3^2}{Q_3^3}\right)=Q_3^{1/4},$$

and identity (4). The cases $\lambda=0$, $\mu=\nu\in\{\frac{1}{3},\frac{1}{4},\frac{1}{6}\}$ are not that beautiful since in these cases automorphic functions y_0-y_1 and y_2-y_1 cannot be rationally expressed in terms of the Ramanujan functions.

Proposition 8. *The following identities hold*:

$$\sum_{n=0}^{\infty}\frac{(3n)!}{(n!)^3}\left(\frac{Q_3^{3/2}-R_3}{3^3\cdot 2Q_3^{3/2}}\right)^n=Q_3^{1/4},\qquad \sum_{n=0}^{\infty}\frac{(4n)!}{(n!)^4}\left(\frac{Q_2^3-R_2^2}{4^4\cdot Q_2^3}\right)^n=Q_2^{1/2},\qquad (24)$$

where the functions Q_2, R_2 *and* Q_3, R_3 *are defined by* (21) *and* (22).

Proof. The first identity in (24) follows from the first one in (23); by the corollary of the classical formula

$$\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n(\frac{2}{4})_n(\frac{3}{4})_n}{(n!)^3}z^n=F^2\left(\frac{1}{8},\frac{3}{8};1;z\right)$$

(see, e.g., [15, Chapter 14, Example 11]) and the second identity in (23) we derive the second relation in (24).

It would be nice if we could continue the sequence (24) by an identity for the series

$$\sum_{n=0}^{\infty}\frac{(5n)!}{(n!)^5}z^n,\qquad (25)$$

all the more so that the substitution

$$z=z(q)=q-770q^2+171525q^3-81623000q^4-35423171250q^5$$
$$-54572818340154q^6-71982448083391590q^7+O(q^8)$$

for (25) is well known (see, e.g., [8, 6]). However the precise expressions of $z(q)$ and of the sum (25) with $z=z(q)$ through classical functions (say, the Ramanujan functions) are not yet known...

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