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BRIEF COMMUNICATIONS

A Third-Order Apéry-Like Recursion for $\zeta(5)$

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In 1978, R. Apéry [1, 2] gave sequences of rational approximations to $\zeta(2)$ and $\zeta(3)$ yielding the irrationality of each of these numbers. One of the key ingredient of Apéry's proof are second-order difference equations with polynomial coefficients satisfied by numerators and denominators of the above approximations.

Recently, V. N. Sorokin [3] and this author [4, 5] independently obtained a similar second-order difference equation for $\zeta(4)$. This recursion does not give Diophantine approximations to $\zeta(4)=\pi^4/90$ proving its irrationality, however it presents an algorithm for fast computation of this constant. The aim of this note is to highlight a possible generalization of the above results for the number $\zeta(5)$, the irrationality of which has not yet been proved.

1. STATEMENT OF THE MAIN RESULT

Consider the difference equation

$$(n+1)^6a_0(n)q_{n+1}+a_1(n)q_n-4(2n-1)a_2(n)q_{n-1}-4(n-1)^4(2n-1)(2n-3)a_0(n+1)q_{n-2}=0, \quad (1)$$

where

$$\begin{aligned}
a_0(n)&=41218n^3-48459n^2+20010n-2871,\\
a_1(n)&=2(48802112n^9+89030880n^8+36002654n^7-24317344n^6-19538418n^5\\
&\qquad+1311365n^4+3790503n^3+460056n^2-271701n-60291),\\
a_2(n)&=3874492n^8-2617900n^7-3144314n^6+2947148n^5+647130n^4\\
&\qquad-1182926n^3+115771n^2+170716n-44541,
\end{aligned}$$

and define its three linearly independent solutions $\{q_n\}$, $\{p_n\}$, and $\{\tilde p_n\}$ by the initial data

$$q_0=-1,\quad q_1=42,\quad q_2=-17934,\qquad p_0=0,\quad p_1=\frac{87}{2},\quad p_2=-\frac{1190161}{64},$$
$$\tilde p_0=0,\quad \tilde p_1=\frac{101}{2},\quad \tilde p_2=-\frac{344923}{16}$$

(here and below, the symbol $\{x_n\}$ stands for the sequence $\{x_n\}_{n=0}^\infty=\{x_0,x_1,x_2,\dots\}$).

Theorem 1. *The sequences*

$$\ell_n = q_n\zeta(5) - p_n, \quad \tilde{\ell}_n = q_n\zeta(3) - \tilde{p}_n, \qquad n = 0, 1, 2, \dots, \tag{2}$$

that also satisfy the difference equation (1), *are of constant sign*:

$$\ell_n > 0, \quad \tilde{\ell}_n < 0, \qquad n = 1, 2, \dots, \tag{3}$$

and the following limit relations hold:

$$\begin{gathered} \lim_{n\to\infty} \frac{\log|\ell_n|}{n} = \lim_{n\to\infty} \frac{\log|\tilde{\ell}_n|}{n} = \log|\mu_2| = -1.08607936\dots, \\ \lim_{n\to\infty} \frac{\log|q_n|}{n} = \lim_{n\to\infty} \frac{\log|p_n|}{n} = \lim_{n\to\infty} \frac{\log|\tilde{p}_n|}{n} = \log|\mu_3|, \end{gathered} \tag{4}$$

where

$$\mu_1 = -0.02001512\dots, \qquad \mu_2 = 0.33753726\dots, \qquad \mu_3 = -2368.31752213\dots \tag{5}$$

are the roots of the characteristic polynomial $\mu^3 + 2368\mu^2 - 752\mu - 16$ *of the recursion* (1).

Note that the sequences $\{q_n\}$, $\{p_n\}$, and $\{\tilde{p}_n\}$ are of alternating sign:

$$(-1)^{n-1}q_n > 0, \quad (-1)^{n-1}p_n > 0, \quad (-1)^{n-1}\tilde{p}_n > 0, \qquad n = 1, 2, \dots.$$

Theorem 1 gives an algorithm for fast computation of the number $\zeta(5)$. Namely, the sequence of rational numbers p_n/q_n converges to $\zeta(5)$ with speed $|\mu_2/\mu_3| < 1.42521964 \cdot 10^{-4}$ (see the table).

n	p_n/q_n	$\lvert\zeta(5) - p_n/q_n\rvert$
0	0	$1.036927755\dots$
1	$\frac{29}{28}$	$0.001213469\dots$
2	$\frac{24289}{23424}$	$0.000000182\dots$
3	$\frac{7682021239}{7408444032}$	$< 2.80 \cdot 10^{-11}$
4	$\frac{24943788950905}{24055474286592}$	$< 4.13 \cdot 10^{-15}$
5	$\frac{81875586674776013003}{78959779279372800000}$	$< 6.02 \cdot 10^{-19}$
6	$\frac{28265375611268633697 5107}{27258770411985496320 0000}$	$< 8.71 \cdot 10^{-23}$
7	$\frac{215903781003833520407770175189}{208214873150908926517286400000}$	$< 1.26 \cdot 10^{-26}$
10		$< 3.71 \cdot 10^{-38}$
20		$< 1.32 \cdot 10^{-76}$
50		$< 5.52 \cdot 10^{-192}$

Further construction of the above solutions of the difference equation (1) leads to the inclusions

$$4D_n^2 q_n \in \mathbb{Z}, \quad 4D_n^7 p_n \in \mathbb{Z}, \quad 4D_n^5 \tilde{p}_n \in \mathbb{Z}, \qquad n = 1, 2, \dots$$

(see (14) and (15)), where D_n denotes the least common multiple of the numbers $1, 2, \dots, n$, while a straightforward verification on the basis of the recursion (1) shows that

$$q_n \in \mathbb{Z}, \quad 2D_n^5 p_n \in \mathbb{Z}, \quad 2D_n^3 \tilde{p}_n \in \mathbb{Z}, \qquad n = 1, 2, \dots. \tag{6}$$

The inclusions (6) do not allow one to prove the irrationality of the number $\zeta(5)$; hence we do not accent our attention on the arithmetic properties of the sequences $\{q_n\}$, $\{p_n\}$, $\{\tilde{p}_n\}$ and only stress the interest of Theorem 1 for applications.

2. AUXILIARY RECURSIONS

The very-well-posed hypergeometric series

$$\begin{aligned} r_n &= \quad n!^4 \sum_{k=1}^{\infty}\Bigl(k+\frac{n}{2}\Bigr)\frac{\prod_{j=1}^{n}(k-j)\cdot\prod_{j=1}^{n}(k+n+j)}{\prod_{j=0}^{n}(k+j)^6}, \\ \tilde r_n &= -n!^4 \sum_{k=1}^{\infty}\Bigl(k+\frac{n}{2}\Bigr)\frac{\prod_{j=0}^{n}(k-j)\cdot\prod_{j=0}^{n}(k+n+j)}{\prod_{j=0}^{n}(k+j)^6}, \end{aligned} \qquad n=0,1,2,\dots, \tag{7}$$

are $\mathbb{Q}$-linear forms in $1, \zeta(3), \zeta(5)$:

$$r_n = u_n\zeta(5)+w_n\zeta(3)-v_n, \quad \tilde r_n = \tilde u_n\zeta(5)+\tilde w_n\zeta(3)-\tilde v_n, \qquad n=0,1,2,\dots.$$

An easy verification shows that

$$\begin{aligned} &r_0=\zeta(5), \qquad r_1 = 9\zeta(5)+33\zeta(3)-49, \qquad r_2 = 469\zeta(5)+\frac{6125}{4}\zeta(3)-\frac{74463}{32}, \\ &\tilde r_0=\zeta(3), \qquad \tilde r_1 = 2\zeta(5)+12\zeta(3)-\frac{33}{2}, \qquad \tilde r_2 = 552\zeta(5)+1764\zeta(3)-\frac{43085}{16}. \end{aligned}$$

Applying the algorithm of creative telescoping [6, Chap. 6] to the series (7) in the style of [4, 5, 7], we arrive at the difference equations

$$n(n+1)^5 b_0(n-1)u_{n+1} - 2nb_1(n)u_n - b_2(n)u_{n-1} + 2(n-1)^5(2n-1)b_0(n)u_{n-2} = 0, \tag{8}$$

where

$$b_0(n) = -a_0(-n), \qquad b_1(n) = a_2(-n), \qquad b_2(n) = -a_1(-n),$$

and

$$n^3(n+1)^3\tilde b_0(n-1)u_{n+1} - 2n\tilde b_1(n)u_n - \tilde b_2(n)u_{n-1} + 2n(n-1)^4(2n-3)\tilde b_0(n)\tilde u_{n-2} = 0, \tag{9}$$

where

$$\begin{aligned} \tilde b_0(n) &= 41218n^7+35648n^6-932n^5-13190n^4-5128n^3+811n^2+957n+174, \\ \tilde b_1(n) &= 3874492n^{12}-14084302n^{11}+12425954n^{10}+8641603n^9-15230839n^8-1369195n^7 \\ &\qquad + 8618417n^6-623249n^5-2785973n^4+308165n^3+495325n^2-40670n-37632, \\ \tilde b_2(n) &= 2(48802112n^{13}-201803328n^{12}+267014032n^{11}-69927236n^{10} \\ &\qquad - 95912858n^9+37524471n^8+30257812n^7-9523224n^6-8524312n^5 \\ &\qquad + 2138687n^4+1507490n^3-398634n^2-111012n+33408). \end{aligned}$$

Note that each of the recursions (8) and (9) has the same characteristic polynomial $\lambda^3-188\lambda^2-2368\lambda+4$; its roots $\lambda_1, \lambda_2, \lambda_3$ ordered in increasing order of their moduli are related to the roots (5) as follows:

$$\mu_1=\lambda_1\lambda_2, \qquad \mu_2=\lambda_1\lambda_3, \qquad \mu_3=\lambda_2\lambda_3.$$

Theorem 2. *The sequences of the linear forms $\{r_n\}$ and their coefficients $\{u_n\}$, $\{w_n\}$, $\{v_n\}$ satisfy the difference equation* (8). *In addition, the inequalities*

$$r_n>0, \quad u_n>0, \quad w_n>0, \quad v_n>0, \qquad n=1,2,\dots, \tag{10}$$

and the limit relations

$$\lim_{n\to\infty}\frac{\log|r_n|}{n} = \log\lambda_1, \qquad \lim_{n\to\infty}\frac{\log|u_n|}{n} = \lim_{n\to\infty}\frac{\log|w_n|}{n} = \lim_{n\to\infty}\frac{\log|v_n|}{n} = \log\lambda_3 \tag{11}$$

hold.

Theorem 3. *The sequences of the linear forms* $\{\tilde r_n\}$ *and their coefficients* $\{\tilde u_n\}$, $\{\tilde w_n\}$, $\{\tilde v_n\}$ *satisfy the difference equation* (9). *In addition, the inequalities*

$$\tilde r_n < 0, \quad \tilde u_n > 0, \quad \tilde w_n > 0, \quad \tilde v_n > 0, \qquad n = 1, 2, \dots, \tag{12}$$

and the limit relations

$$\lim_{n\to\infty} \frac{\log|\tilde r_n|}{n} = \log\lambda_1, \qquad \lim_{n\to\infty} \frac{\log|\tilde u_n|}{n} = \lim_{n\to\infty} \frac{\log|\tilde w_n|}{n} = \lim_{n\to\infty} \frac{\log|\tilde v_n|}{n} = \log\lambda_3 \tag{13}$$

hold.

We mention that Vasilyev's result [8] and our Theorem [4] (see also [9]) on the coincidence of very-well-posed hypergeometric series and a suitable generalization of Beukers' integral [10] lead one to the inclusions

$$\begin{aligned} &2u_n \in \mathbb{Z}, \quad 2D_n^2 w_n \in \mathbb{Z}, \quad 2D_n^5 v_n \in \mathbb{Z}, \\ &2\tilde u_n \in \mathbb{Z}, \quad 2D_n^2 \tilde w_n \in \mathbb{Z}, \quad 2D_n^5 \tilde v_n \in \mathbb{Z}, \end{aligned} \qquad n = 1, 2, \dots . \tag{14}$$

Finally, the required sequences (2) are determined by the formulas

$$\begin{aligned} \ell_n &= \tilde w_n r_n - w_n \tilde r_n = (u_n \tilde w_n - \tilde u_n w_n)\zeta(5) - (\tilde w_n v_n - w_n \tilde v_n), \\ \tilde\ell_n &= u_n \tilde r_n - \tilde u_n r_n = (u_n \tilde w_n - \tilde u_n w_n)\zeta(3) - (u_n \tilde v_n - \tilde u_n v_n), \end{aligned} \qquad n = 0, 1, 2, \dots ;$$

hence

$$q_n = u_n \tilde w_n - \tilde u_n w_n, \quad p_n = \tilde w_n v_n - w_n \tilde v_n, \quad \tilde p_n = u_n \tilde v_n - \tilde u_n v_n, \qquad n = 0, 1, 2, \dots . \tag{15}$$

In order to prove the inequalities (3) and the limit relations (4), it remains to use the estimates (10), (12), relations (11), (13), and Poincaré's theorem.

3. CONCLUDING REMARKS

Recently, another algorithm for the fast computation of $\zeta(5)$ and $\zeta(3)$ (which also does not produce sufficiently good Diophantine approximations to these constants), based on the infinite matrix product

$$\prod_{n=1}^{\infty} \begin{pmatrix} -\dfrac{n}{2(2n+1)} & \dfrac{1}{2n(2n+1)} & \dfrac{1}{n^4} \\ 0 & -\dfrac{n}{2(2n+1)} & \dfrac{5}{4n^2} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \zeta(5) \\ 0 & 0 & \zeta(3) \\ 0 & 0 & 1 \end{pmatrix},$$

was suggested by R. W. Gosper [11]. However the letters [11] do not contain any description of analytic and arithmetic properties of the rational approximations so constructed. Let us also mention an algorithm due to E. A. Karatsuba [12] for the computation of the Riemann zeta function at positive integers.

The above scheme of Sec. 2 also allows one to construct a third-order recursion for simultaneous rational approximations to $\zeta(2)$ and $\zeta(3)$. Namely, consider the difference equation

$$(n+1)^4 \tilde a_0(n) q_{n+1} - \tilde a_1(n) q_n + 4(2n-1)\tilde a_2(n) q_{n-1} - 4(n-1)^2(2n-1)(2n-3)\tilde a_0(n+1) q_{n-2} = 0, \tag{16}$$

where

$$\begin{aligned} \tilde a_0(n) &= 946n^2 - 731n + 153, \\ \tilde a_1(n) &= 2(104060n^6 + 127710n^5 + 12788n^4 - 34525n^3 - 8482n^2 + 3298n + 1071), \\ \tilde a_2(n) &= 3784n^5 - 1032n^4 - 1925n^3 + 853n^2 + 328n - 184, \end{aligned}$$

and define its three linearly independent solutions $\{q'_n\}$, $\{p'_n\}$, and $\{\tilde p'_n\}$ by the initial data

$$q'_0 = 1, \quad q'_1 = 14, \quad q'_2 = 978, \qquad p'_0 = 0, \quad p'_1 = 17, \quad p'_2 = 9405/8,$$
$$\tilde p'_0 = 0, \quad \tilde p'_1 = 23, \quad \tilde p'_2 = 6435/4.$$

Theorem 4. *The sequences* $\{q'_n\}$, $\{p'_n\}$, *and* $\{\tilde{p}'_n\}$ (*of positive sign*) *as well as the sequences*

$$\ell'_n = q'_n\zeta(3) - p'_n, \quad \tilde{\ell}'_n = q'_n\zeta(2) - \tilde{p}'_n, \qquad n = 0, 1, 2, \dots,$$

satisfy the limit relations

$$\lim_{n\to\infty} \frac{\log|\ell'_n|}{n} = \lim_{n\to\infty} \frac{\log|\tilde{\ell}'_n|}{n} = \log|\mu_2| = -1.31018925\dots,$$
$$\lim_{n\to\infty} \frac{\log|q'_n|}{n} = \lim_{n\to\infty} \frac{\log|p'_n|}{n} = \lim_{n\to\infty} \frac{\log|\tilde{p}'_n|}{n} = \log|\mu_3|,$$

where

$$\mu_{1,2} = 0.07260980\dots \pm i0.25981363\dots, \qquad \mu_3 = 219.85478039\dots$$

are the roots of the characteristic polynomial $\mu^3 - 220\mu^2 + 32\mu - 16$ *of the recursion* (16).

In this case, the auxiliary recursions are satisfied by the hypergeometric series

$$r'_n = -n!^2 \sum_{k=1}^{\infty} \frac{\prod_{j=1}^{n}(k-j)}{\prod_{j=0}^{n}(k+j)^3}, \quad \tilde{r}'_n = n!^2 \sum_{k=1}^{\infty} \frac{\prod_{j=0}^{n}(k-j)}{\prod_{j=0}^{n}(k+j)^3}, \qquad n = 0, 1, 2, \dots,$$

which are $\mathbb{Q}$-linear forms in 1, $\zeta(2)$, and $\zeta(3)$.

Similarly to the case of the difference equation (1), explicit computations lead to the inclusions

$$q'_n \in \mathbb{Z}, \quad D_n^3 p'_n \in \mathbb{Z}, \quad D_n^2 \tilde{p}'_n \in \mathbb{Z}, \qquad n = 1, 2, \dots,$$

which are much better than one would expect.

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