One of the numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational

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In this paper we establish the following result.

Theorem. *At least one of the four numbers* $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ *is irrational.*

In the proof we use a generalization of the construction proposed by Rivoal in [1] of linear approximating forms in values of the Riemann ζ-function at the odd points. Namely, by using this analytical construction, it was proved in [1] that there are infinitely many irrational numbers among $\zeta(3), \zeta(5), \zeta(7), \dots$. The irrationality of one of the nine numbers $\zeta(5), \zeta(7), \dots, \zeta(21)$ was proved independently by Rivoal [2] and the author [3]. We also recall that the irrationality of $\zeta(3)$ was established by Apéry [4].

We fix odd numbers q and r, $q \geq r+4$, and a tuple $\eta_0, \eta_1, \eta_2, \dots, \eta_q$ of positive integer parameters satisfying the conditions $\eta_1 \leqslant \eta_2 \leqslant \dots \leqslant \eta_q < \eta_0/2$ and

$$\eta_1 + \eta_2 + \dots + \eta_q \leqslant \eta_0 \cdot \frac{q-r}{2}. \tag{1}$$

For every integer $n > 0$ we define the integer tuple

$$h_0 = \eta_0 n + 2, \qquad h_j = \eta_j n + 1, \quad j = 1, \dots, q,$$

and consider the *rational* function

$$R_n(t) := (h_0 + 2t) \cdot \prod_{j=1}^{r} \frac{1}{(h_j-1)!} \frac{\Gamma(h_j + t)}{\Gamma(1+t)} \cdot \prod_{j=1}^{r} \frac{1}{(h_j-1)!} \frac{\Gamma(h_0 + t)}{\Gamma(1 + h_0 - h_j + t)}$$
$$\times \prod_{j=r+1}^{q} (h_0 - 2h_j)! \frac{\Gamma(h_j + t)}{\Gamma(1 + h_0 - h_j + t)},$$

and also the corresponding quantity

$$F_n := \frac{1}{(r-1)!} \sum_{t=0}^{\infty} R_n^{(r-1)}(t) \tag{2}$$

(by (1), $R_n(t) = O(t^{-2})$, which guarantees the convergence of the series on the right-hand side of (2)).

We put $m_j = \max\{\eta_r, \eta_0 - 2\eta_{r+1}, \eta_0 - \eta_1 - \eta_{r+j}\}$ for $j = 1, \dots, q-r$ and define the integer

$$\Phi_n := \prod_{\sqrt{\eta_0 n} < p \leqslant m_{q-r} n} p^{\varphi(n/p)},$$

where only primes enter the product and

$$\varphi(x) := \min_{0 \leqslant y < 1} \Biggl(\sum_{j=1}^{r} \bigl(\lfloor y \rfloor + \lfloor \eta_0 x - y \rfloor - \lfloor y - \eta_j x \rfloor - \lfloor (\eta_0 - \eta_j) x - y \rfloor - 2 \lfloor \eta_j x \rfloor \bigr)$$
$$+ \sum_{j=r+1}^{q} \bigl(\lfloor (\eta_0 - 2\eta_j) x \rfloor - \lfloor y - \eta_j x \rfloor - \lfloor (\eta_0 - \eta_j) x - y \rfloor \bigr) \Biggr)$$

is an integer-valued non-negative periodic (with period 1) function. We denote by D_N the least common multiple of $1, 2, \dots, N$.

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Lemma 1. (2) *defines a linear form of* $1, \zeta(r+2), \zeta(r+4), \dots, \zeta(q-2)$ *with rational coefficients; moreover,*

$$D^r_{m_1 n} D_{m_2 n} \cdots D_{m_{q-r} n} \cdot \Phi_n^{-1} \cdot F_n \in \mathbb{Z} + \mathbb{Z}\zeta(r+2) + \mathbb{Z}\zeta(r+4) + \cdots + \mathbb{Z}\zeta(q-2). \tag{3}$$

The asymptotics of Φ_n as $n \to \infty$ can be calculated by using the Chudnovskii–Rukhadze–Hata arithmetic method (see the subtrahend in the definition of the constant C_1 in Lemma 3 stated below). Moreover, by the prime number theorem,

$$\lim_{n\to\infty} \frac{\log D_{m_j n}}{n} = m_j, \qquad j = 1, \dots, q-r.$$

We introduce the auxiliary function

$$\begin{aligned} f_0(\tau) = r\eta_0 \log(\eta_0 - \tau) + \sum_{j=1}^{q} \bigl(\eta_j \log(\tau - \eta_j) - (\eta_0 - \eta_j) \log(\tau - \eta_0 + \eta_j)\bigr) \\ - 2\sum_{j=1}^{r} \eta_j \log \eta_j + \sum_{j=r+1}^{q} (\eta_0 - 2\eta_j) \log(\eta_0 - 2\eta_j), \end{aligned}$$

defined in the τ-plane with the cuts $(-\infty, \eta_0 - \eta_1]$ and $[\eta_0, +\infty)$. The following lemma, which characterizes the growth of the linear forms F_n in the case $r = 3$, can be proved by representing (2) as a complex integral on a line $\operatorname{Re} t = \text{const}$ and subsequently applying to it the asymptotics of the Gamma-function and the saddle-point method.

Lemma 2. *Let* $r = 3$ *and let* τ_0 *be a zero of the polynomial*

$$(\tau - \eta_0)^r (\tau - \eta_1) \cdots (\tau - \eta_q) - \tau^r (\tau - \eta_0 + \eta_1) \cdots (\tau - \eta_0 + \eta_q)$$

with $\operatorname{Im} \tau_0 > 0$ *and maximum possible value of* $\operatorname{Re} \tau_0$. *Assume that* $\operatorname{Re} \tau_0 < \eta_0$ *and* $\operatorname{Im} f_0(\tau_0) \notin \pi\mathbb{Z}$. *Then*

$$\overline{\lim_{n\to\infty}} \frac{\log |F_n|}{n} = \operatorname{Re} f_0(\tau_0).$$

If the sequence of linear forms on the left side of (3) assumes non-zero arbitrarily small values as n increases, then in the case $r = 3$ there are irrational numbers among

$$\zeta(5),\ \zeta(7),\ \dots,\ \zeta(q-4),\ \zeta(q-2). \tag{4}$$

Therefore, the following holds.

Lemma 3. *Suppose that* $r = 3$ *and in the above notation* $C_0 = -\operatorname{Re} f_0(\tau_0)$,

$$C_1 = rm_1 + m_2 + \cdots + m_{q-r} - \left(\int_0^1 \varphi(x)\, \mathrm{d}\psi(x) - \int_0^{1/m_{q-r}} \varphi(x) \frac{\mathrm{d}x}{x^2} \right),$$

where $\psi(x)$ *is the logarithmic derivative of the Gamma-function. If* $C_0 > C_1$, *then at least one of the numbers* (4) *is irrational.*

To prove the theorem, we put $r = 3, q = 13$,

$$\eta_0 = 91, \quad \eta_1 = \eta_2 = \eta_3 = 27, \quad \eta_4 = 29,\ \eta_5 = 30,\ \eta_6 = 31,\ \dots,\ \eta_{12} = 37,\ \eta_{13} = 38.$$

Then $C_0 = 227.58019641\dots$, $C_1 = 226.24944266\dots$, and by Lemma 3 there are irrational numbers among $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$.

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