

BEAUTIFUL FORMULAS

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In this paper I give formulas that I like the most. The choice is very subjective. There are tons of formulas and I have to choose only a few!

The notation throughout the paper.

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient

$[x]$ is the largest integer less than or equal to x

$\pi = 3.14159265\dots$, $e = 2.718281828\dots$ are famous constants

$\gamma = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n\right) = 0.57721566\dots$ is the Euler constant

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is the Riemann zeta function

$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.915965594\dots$ is the Catalan constant.

Entry 1 (*I found it by myself, but it's surely classic*) Let the sequence a_n be defined by $a_1 = 2$, $a_{n+1} = a_n^2 - a_n + 1 = a_1 a_2 \dots a_n + 1$. The following identity holds

$$\sum_{k=1}^{\infty} \frac{1}{a_k} = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \dots = 1.$$

Entry 2 (*R.W. Gosper*)

$$\frac{1}{\ln x} + \frac{1}{1-x} = \sum_{n=1}^{\infty} \frac{2^{-n}}{1+x^{2^{-n}}}.$$

(*R. Johnson*) For $x > 1$

$$\frac{1}{\ln x} = \log_2 a \sum_{n=-\infty}^{\infty} \frac{a^n}{x^{a^n} + 1}.$$

Entry 3 (*classic*)

$$\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}.$$

(*A.C. Dixon*)

$$\sum_k (-1)^k \binom{a+b}{a+k} \binom{a+c}{c+k} \binom{b+c}{b+k} = \frac{(a+b+c)!}{a!b!c!}.$$

(*G.H. Hardy?*) For $\operatorname{Re} x > 1$

$$1 - 3 \frac{x-1}{x+1} + 5 \frac{(x-1)(x-2)}{(x+1)(x+2)} - \dots = 0.$$

Entry 4 (*G. Vacca*)

$$\gamma = \sum_{k=1}^{\infty} (-1)^k \frac{[\log_2 k]}{k}.$$

(*classic?*)

$$\gamma = \int_0^1 \ln(-\ln x) dx.$$

(*M. Hata*)

$$\gamma = \frac{1}{2} + \frac{1}{2} \sum \frac{1}{abcd(a+c)(b+d)},$$

where the sum is over integers (a, b, c, d) such that

$$bc - ad = 1, \quad 1 \leq a < b, \quad 1 \leq c < d.$$

Entry 5 (*L. Euler?*)

$$\sum_{n=2}^{\infty} (-1)^n \zeta(n) = 1.$$

(*L. Euler*)

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n) = \gamma.$$

Entry 6 (*classic?*)

$$\zeta(2) = \frac{\pi^2}{6} = 3 \sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n} n^2},$$

(*M.M. Hjortnaes*)

$$\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\binom{2n}{n} n^3},$$

(*classic?*)

$$\zeta(4) = \frac{\pi^4}{90} = \frac{36}{17} \sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n} n^4}.$$

Entry 7 (*F. Beukers; V.N. Sorokin; G. Rhin*)

$$2\zeta(3) = \int_{[0,1]^3} \frac{dx dy dz}{1 - x + xyz} = \int_{[0,1]^3} \frac{dx dy dz}{(1 - xy)(1 - xyz)} = \int_{[0,1]^3} \frac{z dx dy dz}{(1 - xz)(1 - yz)}.$$

(*W.V. Zudilin; classic?*)

$$G = \frac{1}{2} \int_{[0,1]^2} \frac{dx dy}{1 - x^2 + x^2 y^2} = \int_{[0,1]^2} \frac{dx dy}{2 - x^2 - y^2}.$$

Entry 8 (*classic?*)

$$\frac{1}{1^3} - \frac{1}{2^3} + \frac{1}{4^3} - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{8^3} + \dots = 4\sqrt{3} \frac{\pi^3}{243}.$$

Entry 9 (*L. Euler*)

$$\sum_{n_1 > n_2 > \dots > n_k \geq 1} \frac{1}{n_1^2 n_2^2 \dots n_k^2} = \frac{\pi^{2k}}{(2k+1)!}.$$

(*D.V. Vasilyev*)

$$\sum_{n_1 \geq n_2 \geq \dots \geq n_{k+1} \geq 1} \frac{1}{n_1^2 n_2^2 \dots n_k^2 n_{k+1}} = 2\zeta(2k+1).$$

Entry 10 (*L. Euler*)

$$\sum_{n=1}^{\infty} \frac{1}{2^n n^2} = \frac{1}{2} \zeta(2) - \frac{1}{2} \ln^2 2,$$

(*J. Landen*)

$$\sum_{n=1}^{\infty} \frac{1}{2^n n^3} = \frac{7}{8} \zeta(3) - \frac{1}{2} \zeta(2) \ln 2 + \frac{1}{6} \ln^3 2.$$

Entry 11 (*J. Wallis*)

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)}.$$

(*L. Euler*)

$$\frac{1}{\pi} = \sum_{n=2}^{\infty} \frac{1}{2^n} \tan\left(\frac{\pi}{2^n}\right).$$

(*S. Ramanujan*)

$$\frac{2}{\pi} = 1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1 \cdot 3}{2 \cdot 4}\right)^3 - 13\left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^3 + \dots$$

Entry 12 Let $\sigma_k(n) = \sum_{d|n} d^k$. Then

(*classic?*)

$$\sum_{n=1}^{\infty} \sigma_1(n) e^{-2\pi n} = \frac{\pi - 3}{24\pi}.$$

(*classic?*)

$$\sum_{n=1}^{\infty} \sigma_5(n) e^{-2\pi n} = \frac{1}{504}.$$

Entry 13 (*L. Euler*)

$$\sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{3m^2+m}{2}} = \prod_{n=1}^{\infty} (1 - q^n).$$

(*classic?*)

$$\sum_{m=-\infty}^{\infty} (-1)^m (2m+1) q^{\frac{m^2+m}{2}} = 2 \prod_{n=1}^{\infty} (1 - q^n)^3.$$

(*C. G. Jacobi*)

$$\sum_{m=-\infty}^{\infty} q^{m^2} x^m = \prod_{n=0}^{\infty} \{(1 - q^{2n+2})(1 + q^{2n+1}x)(1 + q^{2n+1}x^{-1})\}.$$

Entry 14 (*classic?*)

$$\exp\left(\sum_{m=1}^{\infty} m^{m-1} \frac{t^m}{m!}\right) = 1 + \sum_{n=1}^{\infty} (n+1)^{n-1} \frac{t^n}{n!}.$$

Entry 15 (*Poisson sum formula in terms of generalized functions*) Let $\delta(x)$ be the standard delta function. Then the following identity holds

$$\sum_{n=-\infty}^{\infty} e^{inx} = 2\pi \sum_{n=-\infty}^{\infty} \delta(x - 2\pi n).$$

(*classic?*) If $\alpha, \beta > 0$ and $\alpha\beta = 2\pi$, then

$$\sqrt{\alpha} \sum_{n=-\infty}^{\infty} e^{-\frac{\alpha^2 n^2}{2}} = \sqrt{\beta} \sum_{n=-\infty}^{\infty} e^{-\frac{\beta^2 n^2}{2}}.$$