

Properties of Coefficients of Certain Linear Forms in Generalized Polylogarithms.

Zlobin S.A.

October 21, 2005

UDK 511.36

Abstract. We study properties of coefficients of a linear form, originating from a multiple integral. As a corollary, we prove Vasilyev's conjecture, connected with the problem of irrationality of the Riemann zeta function at odd integers.

Keywords: irrationality, Riemann zeta function, Vasiliev's conjecture, generalized polylogarithm, multiple integral, linear form, denominators, coefficients estimate.

1 Introduction

Define a generalized polylogarithm by the series

$$\text{Le}_{\vec{s}}(z) = \sum_{n_1 \geq n_2 \geq \dots \geq n_l \geq 1} \frac{z^{n_1}}{n_1^{s_1} n_2^{s_2} \dots n_l^{s_l}}$$

for a vector $\vec{s} = (s_1, \dots, s_l)$, $s_j \in \mathbb{N}$. This series converges when $|z| < 1$. In the paper [1], in connection with approximations of values of generalized polylogarithms, it has been proved the following result. Let a_i, b_i, c_j be integers, satisfying the certain inequalities. Then, the identity

$$S(z) = \int_{[0,1]^m} \frac{\prod_{i=1}^m x_i^{a_i-1} (1-x_i)^{b_i-a_i-1}}{\prod_{j=1}^l (1-zx_1x_2\dots x_{r_j})^{c_j}} dx_1 dx_2 \dots dx_m = \sum_{\vec{s}} P_{\vec{s}}(z^{-1}) \text{Le}_{\vec{s}}(z), \quad (1)$$

holds, where $0 = r_0 < r_1 < r_2 < \dots < r_l = m$ and $P_{\vec{s}}$ are polynomials with rational coefficients. This representation is unique because of the linear independence of $\text{Le}_{\vec{s}}(z)$ with different indices over $\mathbb{C}(z)$ (see [1, Corollary 1]).

It is important in arithmetical applications to have estimate for absolute values and a common denominator of coefficients of the polynomials $P_{\vec{s}}$ depending on the parameters a_i, b_i, c_j , as well as on the dimension m of the integral. This is the main aim of this paper.

One of possible applications of integrals of type $S(z)$ is related to the problem of the irrationality of the Riemann zeta function $\zeta(k)$ at odd integers

$k = 3, 5, 7, \dots$. In [2], D.V. Vasilyev considered the integrals

$$V_{m,n} = \int_{[0,1]^m} \frac{\prod_{i=1}^m x_i^n (1-x_i)^n}{(1-x_1(1-x_2(\dots-x_{m-1}(1-x_m)\dots))^{n+1}} dx_1 dx_2 \dots dx_m.$$

He conjectured that

$$V_{2l+1,n} = A_0 + \sum_{j=1}^l A_j \zeta(2j+1), \quad D_n^{2l+1} A_j \in \mathbb{Z}, \quad (2)$$

where D_n is the least common multiple of $1, 2, \dots, n$. The integral $V_{3,n}$ is equal to the integral, which was used by F. Beukers for the proof of the irrationality of $\zeta(3)$ (see [3]). The equality (2) holds for it. Vasilyev proved (2) for $m = 5$. Later W.V. Zudilin ([4]) showed (2) with the weaker inclusion $D_n^{2l+2} \Phi_n^{-1} A_j \in \mathbb{Z}$, where Φ_n is the product of prime numbers $p < n$ for which $2/3 \leq \{n/p\} < 1$ ($\{\cdot\}$ denotes the fractional part of a number). The validity of $D_n^{2l+1} A_j \in \mathbb{Z}$ was proved by C. Krattenthaler and T. Rivoal ([5, Théorème 1]). Their proof is technically complicated. In this paper we prove (2) using the following representation $V_{m,n}$ in the form (1) (see [6, Corollary 2]):

$$V_{2l+1,n} = \int_{[0,1]^{2l+1}} \frac{\prod_{i=1}^{2l+1} x_i^n (1-x_i)^n dx_1 dx_2 \dots dx_{2l+1}}{\prod_{j=1}^l (1-x_1 \dots x_{2j})^{n+1} (1-x_1 x_2 \dots x_{2l} x_{2l+1})^{n+1}}. \quad (3)$$

We prove theorems 1 and 2 in sections 3 and 4 of this article. They give the estimate on the common denominator and the values of the coefficients $P_{\vec{s}}$ in (1) under the certain conditions.

2 Elementary Sums

We call a sum of type

$$\sum_{n_1 \geq n_2 \geq \dots \geq n_l \geq 1} z^{n_1-1} \prod_{j=1}^l \frac{1}{(n_j + p_j)^{u_j}}, \quad p_j \in \{0, 1, 2, \dots\}, \quad u_j \in \mathbb{N}, \quad (4)$$

elementary. From [1, Theorem 1] it follows that this sum can be expressed in the form (1).

In what follows, for any vector $\vec{s} = (s_1, s_2, \dots, s_l)$ we use the notation $w(\vec{s}) = s_1 + s_2 + \dots + s_l$. The *height* of the polynomial is the maximum of the absolute values of its coefficients.

Lemma 1 *Let $P = \max_{1 \leq j \leq l} p_j$. Then, for the sum (4), the heights of the polynomials $P_{\vec{s}}$ do not exceed*

$$\max(l! \cdot (w(\vec{u})2^{w(\vec{u})})^{l-1} P^l, 1); \quad (5)$$

moreover, $D_P^{w(\vec{u})-w(\vec{s})} P_{\vec{s}}(z) \in \mathbb{Z}[z]$.

Proof. We use the following notation: $r_0 = 0$, $r_j = u_1 + u_2 + \dots + u_j$, $m = r_l = w(\vec{u})$. By [1, Lemma 2] it is possible to write expression (4) as the integral

$$I(p_1, p_2, \dots, p_l) = \int_{[0,1]^m} \frac{\prod_{j=1}^l (x_{r_{j-1}+1} x_{r_{j-1}+2} \dots x_{r_j})^{p_j}}{\prod_{j=1}^l (1 - z x_1 x_2 \dots x_{r_j})} dx_1 dx_2 \dots dx_m.$$

We prove Lemma 1 by induction on the vector $(l, p_1 + p_2 + \dots + p_l)$. We order vectors (l, k) in lexicographic ally, i.e.

$$(l_1, k_1) < (l_2, k_2) \Leftrightarrow l_1 < l_2 \text{ or } l_1 = l_2 \text{ and } k_1 < k_2.$$

The statement, which is proved by induction, is a little stricter than the statement of the lemma: the heights of $P_{\vec{s}}(z)$ do not exceed

$$\max \left(\sum_{j=1}^l p_j \cdot (l-1)! \cdot (m2^m P)^{l-1}, 1 \right).$$

This estimate is really more precise than (5) since $\sum_{j=1}^l p_j \leq l \cdot P$. The induction base ($p_1 = p_2 = \dots = p_l = 0$) follows from (4): $I(0, 0, \dots, 0) = z^{-1} \text{Le}_{u_1, u_2, \dots, u_l}(z)$.

Let $p_h > 0$ for some $h > 1$. From the equality

$$\begin{aligned} (x_{r_{h-1}+1} x_{r_{h-1}+2} \dots x_{r_h})^{p_h} &= (x_{r_{h-1}+1} x_{r_{h-1}+2} \dots x_{r_h})^{p_h-1} \\ &\quad + (x_{r_{h-1}+1} x_{r_{h-1}+2} \dots x_{r_h})^{p_h} (1 - z x_1 x_2 \dots x_{r_{h-1}}) \end{aligned}$$

$$-(x_{r_{h-1}+1}x_{r_{h-1}+2}\dots x_{r_h})^{p_h-1}(1-zx_1x_2\dots x_{r_h})$$

it follows that

$$\begin{aligned} I(p_1, p_2, \dots, p_h, \dots, p_l) &= I(p_1, p_2, \dots, p_h - 1, \dots, p_l) \\ &+ \int_{[0,1]^m} \frac{\prod_{j=1}^l (x_{r_{j-1}+1}x_{r_{j-1}+2}\dots x_{r_j})^{p_j}}{\prod_{j \neq h-1}^l (1-zx_1x_2\dots x_{r_j})} dx_1 dx_2 \dots dx_m \\ &- \int_{[0,1]^m} \frac{\prod_{j=1}^l (x_{r_{j-1}+1}x_{r_{j-1}+2}\dots x_{r_j})^{p'_j}}{\prod_{j \neq h}^l (1-zx_1x_2\dots x_{r_j})} dx_1 dx_2 \dots dx_m, \end{aligned}$$

where $p'_j = p_j$ for $j \neq h$ and $p'_h = p_h - 1$. By [1, Lemma 2] we write this equality as

$$\begin{aligned} I(p_1, p_2, \dots, p_h, \dots, p_l) \\ &= I(p_1, p_2, \dots, p_h - 1, \dots, p_l) \end{aligned} \quad (6)$$

$$\begin{aligned} &+ \sum_{n_1 \geq n_2 \geq \dots \geq n_{l-1} \geq 1} z^{n_1-1} \prod_{j=1}^{h-2} \frac{1}{(n_j + p_j)^{u_j}} \\ &\times \frac{1}{(n_{h-1} + p_{h-1})^{u_{h-1}}(n_{h-1} + p_h)^{u_h}} \cdot \prod_{j=h}^{l-1} \frac{1}{(n_j + p_{j+1})^{u_{j+1}}} \end{aligned} \quad (7)$$

$$\begin{aligned} &- \sum_{n_1 \geq n_2 \geq \dots \geq n_{l-1} \geq 1} z^{n_1-1} \prod_{j=1}^{h-1} \frac{1}{(n_j + p_j)^{u_j}} \\ &\times \frac{1}{(n_h + p_h - 1)^{u_h}(n_h + p_{h+1})^{u_{h+1}}} \cdot \prod_{j=h+1}^{l-1} \frac{1}{(n_j + p_{j+1})^{u_{j+1}}} \end{aligned} \quad (8)$$

If $h = l$, the subtracted sum reads as

$$\frac{1}{p_l^{u_l}} \sum_{n_1 \geq n_2 \geq \dots \geq n_{l-1} \geq 1} z^{n_1-1} \prod_{j=1}^{l-1} \frac{1}{(n_j + p_j)^{u_j}}$$

Now we consider in detail the sum (7). If $p_{h-1} = p_h$, then

$$\frac{1}{(n_{h-1} + p_{h-1})^{u_{h-1}}(n_{h-1} + p_h)^{u_h}} = \frac{1}{(n_{h-1} + p_{h-1})^{u_{h-1}+u_h}},$$

i.e. the sum (7) is elementary and it is possible to apply the induction hypothesis to it. In this case the heights of polynomials $P_{\vec{t}}(z)$ in the sum decomposition (into a linear form) do not exceed

$$(l-1)! \cdot (m2^m)^{l-2} P^{l-1},$$

and the common denominator of coefficients of $P_{\vec{t}}(z)$ divides $D_P^{m-w(\vec{t})}$. If $p_{h-1} \neq p_h$, then we take the following partial fraction decomposition:

$$\frac{1}{(n_{h-1} + p_{h-1})^{u_{h-1}}(n_{h-1} + p_h)^{u_h}} = \sum_{k=1}^{u_{h-1}} \frac{A_k}{(n_{h-1} + p_{h-1})^k} + \sum_{k=1}^{u_h} \frac{B_k}{(n_{h-1} + p_h)^k},$$

$$A_k = (-1)^{u_{h-1}-k} \binom{u_{h-1} + u_h - k - 1}{u_{h-1} - k} \frac{1}{(p_h - p_{h-1})^{u_{h-1} + u_h - k}},$$

$$B_k = (-1)^{u_h - k} \binom{u_{h-1} + u_h - k - 1}{u_h - k} \frac{1}{(p_{h-1} - p_h)^{u_{h-1} + u_h - k}}.$$

Substituting this equality into (7), we write (7) as the sum of $u_{h-1} + u_h$ elementary sums (with coefficients A_k and B_k). We can apply the induction hypothesis for each of them. Consider one of them,

$$\sum_{n_1 \geq n_2 \geq \dots \geq n_{l-1} \geq 1} z^{n_1-1} \prod_{j=1}^{h-2} \frac{1}{(n_j + p_j)^{u_j}} \cdot \frac{1}{(n_{h-1} + p_{h-1})^k} \cdot \prod_{j=h}^{l-1} \frac{1}{(n_j + p_{j+1})^{u_{j+1}}}.$$

The corresponding parameters in it are

$$l' = l - 1, \quad m' = m + k - u_{h-1} - u_h, \quad \vec{p}' = (p_1, \dots, p_{h-2}, p_{h-1}, p_{h+1}, \dots, p_l).$$

Let $P_{\vec{t}}(z)$ be the polynomials in the decomposition into a linear form in generalized polylogarithms; then the common denominator of the coefficients of $P_{\vec{t}}(z)$ divides $D_P^{m'-w(\vec{t})}$. Since $D_P^{u_{h-1}+u_h-k} A_k \in \mathbb{Z}$, we have $D_P^{m-w(\vec{t})}(A_k \cdot P_{\vec{t}}(z)) \in \mathbb{Z}[z]$ as required. The heights of $P_{\vec{t}}(z)$ do not exceed

$$(l-1)! \cdot (m2^m)^{l-2} \cdot P^{l-1}.$$

Consequently, the heights of the polynomials in the decomposition of sum (7) do not exceed

$$\left(\sum_{k=1}^{u_{h-1}} |A_k| + \sum_{k=1}^{u_h} |B_k| \right) \cdot (l-1)! \cdot (m2^m)^{l-2} \cdot P^{l-1}$$

$$\begin{aligned}
&\leq \left(\sum_{k=1}^{u_{h-1}} \binom{u_{h-1} + u_h - k - 1}{u_{h-1} - k} + \sum_{k=1}^{u_h} \binom{u_{h-1} + u_h - k - 1}{u_h - k} \right) \\
&\quad \times (l-1)! \cdot (m2^m)^{l-2} \cdot P^{l-1} \\
&\leq (u_{h-1} + u_h) 2^{u_{h-1} + u_h - 2} \cdot (l-1)! \cdot (m2^m)^{l-2} \cdot P^{l-1} \\
&\leq m2^{m-2} \cdot (l-1)! \cdot (m2^m)^{l-2} \cdot P^{l-1} \\
&\leq \frac{1}{2} \cdot (l-1)! \cdot (m2^m P)^{l-1}.
\end{aligned}$$

Sum (8) is considered similarly. Further, we can apply the induction hypothesis to the integral $I(p_1, p_2, \dots, p_{h-1}, \dots, p_l)$. For all three summands (6), (7), (8), denominators of the coefficients of the polynomial coefficients of $\text{Le}_t^{\vec{z}}(z)$ in the linear form (1) divide $D_P^{m-w(\vec{t})}$. The heights of the polynomials $P_{\vec{s}}(z)$ for the initial sum, in case of $\sum_{j=1}^l p_j > 1$, do not exceed

$$\begin{aligned}
&\left(\sum_{j=1}^l p_j - 1 \right) \cdot (l-1)! \cdot (m2^m P)^{l-1} + 2 \cdot \frac{1}{2} \cdot (l-1)! \cdot (m2^m P)^{l-1} \\
&= \sum_{j=1}^l p_j \cdot (l-1)! \cdot (m2^m P)^{l-1}.
\end{aligned}$$

If $\sum_{j=1}^l p_j = 1$, vectors of the generalized polylogarithms from the decomposition of (7) and (8) have length less than l , and in the decomposition $I(\{0\}_l)$ there is exactly one polylogarithm of length l , i.e. the sets of the polylogarithms are not intersected and the estimate on the heights in this case is also valid.

It remains to prove the statement of lemma for the integral

$$I(p_1, 0, \dots, 0) = \int_{[0,1]^m} \frac{(x_1 x_2 \dots x_{r_1})^{p_1}}{\prod_{j=1}^l (1 - z x_1 x_2 \dots x_{r_j})} dx_1 dx_2 \dots dx_m.$$

From the equality

$$(x_1 x_2 \dots x_{r_1})^{p_1} = z^{-1} (x_1 x_2 \dots x_{r_1})^{p_1-1} - z^{-1} (x_1 x_2 \dots x_{r_1})^{p_1-1} (1 - z x_1 x_2 \dots x_{r_1})$$

it follows that

$$I(p_1, 0, \dots, 0) = z^{-1} I(p_1 - 1, 0, \dots, 0)$$

$$\begin{aligned}
& - z^{-1} \int_{[0,1]^m} \frac{(x_1 x_2 \dots x_{r_1})^{p_1-1}}{\prod_{j=2}^l (1 - z x_1 x_2 \dots x_{r_j})} dx_1 dx_2 \dots dx_m \\
& = z^{-1} I(p_1 - 1, 0, \dots, 0) \\
& - z^{-1} \sum_{n_1 \geq \dots \geq n_{l-1} \geq 1} z^{n_1-1} \frac{1}{(n_1 + p_1 - 1)^{u_1} n_1^{u_2}} \prod_{j=2}^{l-1} \frac{1}{n_j^{u_{j+1}}},
\end{aligned}$$

Thus, one can proceed as before in the case $p_h > 0$ for $h > 1$. Now the lemma is completely proved.

3 Denominators of linear form coefficients

Let us study denominators of the coefficients of the linear form. We shall use the notion of integer-valued polynomial. For a polynomial of degree N to be integer-valued it is sufficient that it possesses integer values at $N + 1$ neighbour integer points (see [8, Theorem 12.1]).

Let Δ be a fixed nonnegative integer. We say that a rational function $R(x)$ is Δ -normal if it can be represented as

$$R(x) = \sum_{\alpha \in \mathcal{A}} \sum_{m=1}^M \frac{A_{m,\alpha}}{(x + \alpha)^m} + P(x),$$

where \mathcal{A} is a set of nonnegative integers from a certain segment $[\alpha_1, \alpha_2]$, $D_{\Delta}^{M-m} A_{m,\alpha} \in \mathbb{Z}$ and $D_{\Delta}^M P(x)$ is an integer-valued polynomial.

Lemma 2 *Multiplying Δ -normal function by an integer-valued polynomial of degree $\leq \Delta$ remains it Δ -normal.*

Proof. An integer-valued polynomial $D_{\Delta}^M P(x)$, multiplied by another integer-valued polynomial, remains the integer-valued. The statement of the lemma would be proved if we demonstrate it for

$$R(x) = \frac{A_{m,\alpha}}{(x + \alpha)^m}, \quad D_{\Delta}^{M-m} A_{m,\alpha} \in \mathbb{Z}.$$

It is carried out by induction on m . We check firstly the induction base $m = 1$.

Let $T(x)$ be an integer-valued polynomial of degree $\leq \Delta$ and α be an integer. Then

$$\frac{T(x)}{x + \alpha} = \frac{T(-\alpha)}{x + \alpha} + Q(x),$$

where $Q(x)$ is a polynomial of degree $\leq \Delta - 1$ (if $\Delta = 0$ it is absent). By the hypothesis, $T(-\alpha)$ is an integer. Consider $Q(x)$ at the points $x = -\alpha + k$, where $k = 1, 2, \dots, \Delta$:

$$Q(-\alpha + k) = \frac{T(-\alpha + k) - T(-\alpha)}{k}.$$

Multiplying all these numbers by D_Δ gives integers, hence $D_\Delta Q(x)$ is an integer-valued polynomial.

Thus, if $m = 1$,

$$R(x)T(x) = \frac{A_{1,\alpha}T(-\alpha)}{x + \alpha} + A_{1,\alpha}Q(x).$$

In addition,

$$D_\Delta^{M-1}(A_{1,\alpha} \cdot T(-\alpha)) = (D_\Delta^{M-1} \cdot A_{1,\alpha})T(-\alpha) \in \mathbb{Z}$$

and

$$D_\Delta^M(A_{1,\alpha} \cdot Q(x)) = (D_\Delta^{M-1} \cdot A_{1,\alpha}) \cdot (D_\Delta \cdot Q(x))$$

is an integer-valued polynomial.

Suppose that $m > 1$. Then

$$R(x)T(x) = \frac{A_{m,\alpha}}{(x + \alpha)^{m-1}} \cdot \frac{T(x)}{x + \alpha} = \frac{A_{m,\alpha}T(-\alpha)}{(x + \alpha)^m} + \frac{A_{m,\alpha}}{(x + \alpha)^{m-1}}Q(x).$$

Since

$$D_\Delta^{M-m} \cdot (A_{m,\alpha}T(-\alpha)) = (D_\Delta^{M-m} \cdot A_{m,\alpha})T(-\alpha) \in \mathbb{Z},$$

the first summand is Δ -normal. Write the second summand as

$$\frac{A_{m,\alpha}/D_\Delta}{(x + \alpha)^{m-1}} \cdot (D_\Delta Q(x)).$$

Since $D_\Delta^{M-(m-1)} \cdot (A_{m,\alpha}/D_\Delta) \in \mathbb{Z}$ and $D_\Delta Q(x)$ is an integer-valued polynomial, we can apply the induction hypothesis to the latter expression. The lemma is proved.

Define the *index* of a rational function $R(x) = \frac{P(x)}{Q(x)}$ as $I(R) = \deg P - \deg Q$.

Lemma 3 *Suppose that for the sum*

$$\mathcal{F} = \sum_{n_1=1}^{\infty} z^{n_1-1} R_1(n_1) \sum_{n_2=1}^{n_1} R_2(n_2) \cdots \sum_{n_l=1}^{n_{l-1}} R_l(n_l),$$

the following inequalities are valid:

$$\sum_{j=1}^{j_1} (I(R_j) + 1) \leq 0, \quad \sum_{j=j_1}^{j_2} (I(R_j) + 1) \leq \Delta \quad (9)$$

for any $1 \leq j_1 \leq j_2 \leq l$, and that the functions R_j are Δ -normal. Then \mathcal{F} is expressed as a finite sum $\sum_i \lambda_i \mathcal{F}_i$, $\lambda_i \in \mathbb{Q}$, where

$$\mathcal{F}_i = \sum_{n_1=1}^{\infty} z^{n_1-1} R_{i,1}(n_1) \sum_{n_2=1}^{n_1} R_{i,2}(n_2) \cdots \sum_{n_{l(i)}=1}^{n_{l(i)-1}} R_{i,l(i)}(n_{l(i)}),$$

and $I(R_{i,j}) < 0$ for any i, j . In addition, the functions $R_{i,j}$ are Δ -normal and $D_{\Delta}^{w_i} \lambda_i \in \mathbb{Z}$, where

$$w_i = \sum_{j=1}^l M_j - \sum_{j=1}^{l(i)} M_{i,j},$$

$M_j, M_{i,j}$ is the maximal order of poles of the functions R_j and $R_{i,j}$.

Proof. We proceed by induction on the vector (l, k) , where k is the number of the functions R_j with $I(R_j) \geq 0$ ($0 \leq k < l$). Order vectors (l, k) in lexicographic ally. The induction base $l = 1$ is clear, since $I(R_1) \leq -1$ in this case by the hypothesis. We prove the statement for a vector (l, k) assuming that for smaller vectors it is already proved. If $k = 0$, it is nothing to prove, since $I(R_j) < 0$ for any j . Let $k > 0$, i.e. there exists j such that $I(R_j) \geq 0$. The condition $I(R_1) \leq -1$ implies $j > 1$. Expressing R_j as the sum of a polynomial and a proper fraction, write \mathcal{F} as the sum of two summands. The summand with the proper fraction (Δ -normal) has the

number k smaller by one, hence we can apply the induction base to it. Now consider the second summand, in which $R_j(x) = P(x)$ is a polynomial. From the normality of R_j , it follows that the polynomial $D_{\Delta}^{M_j} P$ is integer-valued and, in addition, the sum of the maximal orders of poles of the functions R_j is just smaller by M_j compared with \mathcal{F} .

a) If $j = l$ summation of the latter gives

$$\sum_{n_l=1}^{n_{l-1}} P(n_l) = Q(n_{l-1}),$$

where $D_{\Delta}^{M_j} Q$ is an integer-valued polynomial of degree $\deg P + 1$. Thereby, R_{l-1} is multiplied by Q . Thus, compared with the initial sum, the number of summations is decreased by one. We can apply the induction hypothesis to the above sum, multiplied by $D_{\Delta}^{M_j}$, since the index vector of involved rational functions equals $(I(R_1), \dots, I(R_{l-2}), I(R_{l-1}) + I(R_l) + 1)$, and multiplying by $D_{\Delta}^{M_j} Q(x)$ of the function R_{l-1} remains it P -normal by Lemma 2, because of condition (9) for $j_1 = j_2 = j$ we have

$$\deg Q(x) = \deg P + 1 = I(R_j) + 1 \leq \Delta.$$

b) Suppose that $R_j(x) = P(x)$ for $1 < j < l$. Write the initial sum as

$$\sum_{n_1=1}^{\infty} z^{n_1-1} R_1(n_1) \sum_{n_2=1}^{n_1} R_2(n_2) \cdots \sum_{n_{j-1}=1}^{n_{j-2}} R_{j-1}(n_{j-1}) \sum_{n_j=1}^{n_{j-1}} P(n_j) \sum_{n_{j+1}=1}^{n_j} f(n_{j+1}),$$

where

$$f(n_{j+1}) = R(n_{j+1}) \sum_{n_{j+2}=1}^{n_{j+1}} R_{j+2}(n_{j+2}) \cdots \sum_{n_l=1}^{n_{l-1}} R_l(n_l)$$

We have equalities:

$$\begin{aligned} & \sum_{n_j=1}^{n_{j-1}} P(n_j) \sum_{n_{j+1}=1}^{n_j} f(n_{j+1}) \\ &= \sum_{n_j=1}^{n_{j-1}} P(n_j) \sum_{n_{j+1}=1}^{n_{j-1}} f(n_{j+1}) - \sum_{n_j=1}^{n_{j-1}} P(n_j) \sum_{n_{j+1}=n_j+1}^{n_{j-1}} f(n_{j+1}) \end{aligned}$$

$$\begin{aligned}
&= Q_1(n_{j-1}) \sum_{n_{j+1}=1}^{n_{j-1}} f(n_{j+1}) - \sum_{n_{j+1}=2}^{n_{j-1}} f(n_{j+1}) \sum_{n_j=1}^{n_{j+1}-1} P(n_j) \\
&= Q_1(n_{j-1}) \sum_{n_{j+1}=1}^{n_{j-1}} f(n_{j+1}) - \sum_{n_{j+1}=1}^{n_{j-1}} Q_2(n_{j+1}) f(n_{j+1});
\end{aligned}$$

in addition, $\deg Q_1 = \deg Q_2 = \deg P + 1$, $Q_2(1) = 0$. Thereby, we express the initial sum as the difference of sums with a smaller repetition factor, corresponding to the vectors

$$(I(R_1), \dots, I(R_{j-1}) + I(R_j) + 1, I(R_{j+1}), \dots, I(R_l)),$$

$$(I(R_1), \dots, I(R_{j-1}), I(R_{j+1}) + I(R_j) + 1, \dots, I(R_l)).$$

The inequality (9) is also valid for each sum. Since $D_{\Delta}^{M_j} P(x)$ is an integer-valued polynomial and $Q_1(x)$, $Q_2(x)$ are polynomials such that $Q_1(n) = \sum_{k=1}^n P(k)$, $Q_2(n) = \sum_{k=1}^{n-1} P(k)$ for every integer $n \geq 1$, it follows that $D_{\Delta}^{M_j} Q_1$ and $D_{\Delta}^{M_j} Q_2$ are integer-valued. Multiplying functions R_{j-1} and R_{j+1} by $D_{\Delta}^{M_j} Q_1(x)$ and $D_{\Delta}^{M_j} Q_2(x)$ remains them P -normal by Lemma 2, because of

$$\deg Q_1(x) = \deg Q_2(x) = \deg P + 1 = I(R_j) + 1 \leq \Delta.$$

The last inequality is due to condition (9) for $j_1 = j_2 = j$. Thus, we can apply the induction hypothesis for each of two sums, multiplied by $D_{\Delta}^{M_j}$. This completes the proof.

Lemma 4 *Let the parameters a_i , b_i , c_j be integer, $b_i > a_i \geq 1$ for $i = 1, \dots, m$, $P = \max_{1 \leq i \leq m} b_i - 2$, $q_j = \sum_{i=r_{j-1}+1}^{r_j} (b_i - a_i)$ and let inequalities $1 \leq c_j \leq P+1$, $c_1 + \dots + c_j \leq q_1 + \dots + q_j$, $j = 1, \dots, l$; $c_{j_1-1} + \sum_{j=j_1}^{j_2} (c_j - q_j) \leq P+1$, $1 < j_1 \leq j_2 \leq l$ be valid. Suppose that $P_{\vec{s}}$ are the polynomials from the linear form $S(z) = \sum_{\vec{s}} P_{\vec{s}}(z^{-1}) \text{Le}_{\vec{s}}(z)$. Then the polynomial $D_P^{m-w(\vec{s})} P_{\vec{s}}(z)$ has integer coefficients.*

Proof. Using [1, Lemma 2], represent the integral $S(z)$ as

$$S(z) = \frac{\prod_{i=1}^m \Gamma(b_i - a_i)}{\prod_{j=1}^l \Gamma(c_j)} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{n_1} \dots \sum_{n_l=1}^{n_{l-1}} z^{n_1-1}$$

$$\times \frac{\prod_{j=1}^l [(n_j - n_{j+1} + 1)(n_j - n_{j+1} + 2) \cdots (n_j - n_{j+1} + c_j - 1)]}{\prod_{j=1}^l \prod_{i=r_{j-1}+1}^{r_j} [(n_j + a_i - 1)(n_j + a_i) \cdots (n_j + b_i - 2)]},$$

letting $n_{l+1} \equiv 1$. From the known formula

$$(x - y + 1) \cdots (x - y + n) = \sum_{k=0}^n (-1)^k \binom{n}{k} (x + k + 1) \cdots (x + n) \\ \times (y + 1) \cdots (y + k - 1)$$

(see, for instance, [2, Lemma 5]) it follows that

$$(n_j - n_{j+1} + 1) \cdots (n_j - n_{j+1} + c_j - 1) = \sum_{k_j=0}^{c_j-1} (-1)^{k_j} \binom{c_j-1}{k_j} \\ \times (n_j + k_j + 1)(n_j + k_j + 2) \cdots (n_j + c_j - 1) \cdot n_{j+1}(n_{j+1} + 1) \cdots (n_{j+1} + k_j - 1).$$

Using this equality for each j , we express $S(z)$ as a linear combination with integer coefficients of sums (with fixed k_j) of type

$$\sum_{n_1=1}^{\infty} \sum_{n_2=1}^{n_1} \cdots \sum_{n_l=1}^{n_{l-1}} z^{n_1-1} \prod_{j=1}^l p_j^1(n_j) p_j^2(n_{j+1}) \\ \times \prod_{i=r_{j-1}+1}^{r_j} \frac{\Gamma(b_i - a_i)}{(n_j + a_i - 1)(n_j + a_i) \cdots (n_j + b_i - 2)},$$

where

$$p_j^1(x) = \frac{(x + k_j + 1)(x + k_j + 2) \cdots (x + c_j - 1)}{(c_j - k_j - 1)!}, \\ p_j^2(x) = \frac{x(x + 1) \cdots (x + k_j - 1)}{k_j!}$$

are integer-valued polynomials. Write the last expression as

$$\sum_{n_1=1}^{\infty} z^{n_1-1} R_1(n_1) \sum_{n_2=1}^{n_1} R_2(n_2) \cdots \sum_{n_l=1}^{n_{l-1}} R_l(n_l).$$

where

$$R_j(x) = p_j^1(x) p_{j-1}^2(x) \prod_{i=r_{j-1}+1}^{r_j} \frac{\Gamma(b_i - a_i)}{(x + a_i - 1)(x + a_i) \cdots (x + b_i - 2)}.$$

For $j = 1$, define $p_0^2(x) \equiv 1$, $c_0 = 1$, $k_0 = 0$.

Since $|(b_{i_1} - 2) - (a_{i_2} - 1)| \leq P - (\min_{1 \leq i \leq m} a_i - 1) \leq P$, the product

$$\prod_{i=r_{j-1}+1}^{r_j} \frac{\Gamma(b_i - a_i)}{(x + a_i - 1)(x + a_i) \cdots (x + b_i - 2)}$$

is P -normal. Consequently, we can apply Lemma 2 to it multiplied by $p_j^1(x)$ and $p_{j-1}^2(x)$. The estimate on polynomial degrees are valid: $\deg p_j^1 \leq c_j - 1 \leq P$ and $\deg p_{j-1}^2 \leq c_{j-1} - 1 \leq P$. Thus, R_j is P -normal function.

Verify condition (9) for the function R_j :

$$\begin{aligned} \sum_{j=j_1}^{j_2} (I(R_j) + 1) &= \sum_{j=j_1}^{j_2} (k_{j-1} + (c_j - k_j - 1) - q_j + 1) \\ &\leq c_{j_1-1} - 1 + \sum_{j=j_1}^{j_2} (c_j - q_j) \leq P. \end{aligned}$$

The last inequality holds by the hypothesis of the lemma.

Using Lemma 3, we may assume that $I(R_j) < 0$ for any j , and R_j is P -normal. Equivalently, we expressed $S(z)$ as a linear combination with integer coefficients of sums

$$\sum_{n_1=1}^{\infty} z^{n_1-1} \frac{A_{\vec{u}, \vec{\alpha}}}{(n_1 + \alpha_1)^{u_1}} \sum_{n_2=1}^{n_1} \frac{1}{(n_2 + \alpha_2)^{u_2}} \cdots \sum_{n_{l'}=1}^{n_{l'-1}} \frac{1}{(n_{l'} + \alpha_{l'})^{u_{l'}}},$$

where $l' \leq l$, $p \leq \alpha_j \leq P$. Herewith $D_P^{m-w(\vec{u})} A_{\vec{u}, \vec{\alpha}} \in \mathbb{Z}$. Furthermore, for a polynomial $P_{\vec{s}}$ in the decomposition of the elementary sum

$$\sum_{n_1=1}^{\infty} z^{n_1-1} \frac{1}{(n_1 + \alpha_1)^{u_1}} \sum_{n_2=1}^{n_1} \frac{1}{(n_2 + \alpha_2)^{u_2}} \cdots \sum_{n_{l'}=1}^{n_{l'-1}} \frac{1}{(n_{l'} + \alpha_{l'})^{u_{l'}}},$$

into the linear form $\sum_{\vec{s}} P_{\vec{s}}(z^{-1}) \text{Le}_{\vec{s}}(z)$, we have the inclusion $D_P^{w(\vec{u})-w(\vec{s})} P_{\vec{s}}(z) \in \mathbb{Z}[z]$ by Lemma 1. This implies the lemma.

Remark. Lemma 4 remains valid if some of c_j are equal to zero.

Theorem 1 *Let the parameters a_i, b_i, c_j be integers, $b_i > a_i \geq 1$ for $i = 1, \dots, m$ and $c_j \geq 1, c_1 + \dots + c_j \leq q_1 + \dots + q_j$, where $q_j = \sum_{i=r_{j-1}+1}^{r_j} (b_i - a_i)$, $j = 1, \dots, l$; let d_j be nonnegative integers, satisfying $d_j \leq c_j$ for $j = 1, \dots, l$ and $\sum_{k=j}^l d_k < a_i$ for $j = 1, \dots, l$ and $r_{j-1} < i \leq r_j$. Denote*

$$\Delta = \max_{1 \leq j \leq l} \max_{r_{j-1} < i \leq r_j} (b_i - \sum_{k=j}^l d_k - 2).$$

Assume that the following inequalities are valid: $1 \leq c_j \leq \Delta + 1$, $c_{j_1-1} + \sum_{j=j_1}^{j_2} (c_j - q_j) \leq \Delta + 1$, $1 < j_1 \leq j_2 \leq l$, and that $P_{\vec{s}}$ are the polynomials from the linear form $S(z) = \sum_{\vec{s}} P_{\vec{s}}(z^{-1}) \text{Le}_{\vec{s}}(z)$. Then the polynomial $D_{\Delta}^{m-w(\vec{s})} P_{\vec{s}}(z)$ has integer coefficients.

Proof. Expand the integrand of $S(z)$ using the following equalities:

$$(x_1 x_2 \dots x_{r_j})^{d_j} = \left(\frac{1 - (1 - z x_1 x_2 \dots x_{r_j})}{z} \right)^{d_j}, \quad j = 1, \dots, l.$$

It is possible to do, since $\sum_{k=j}^l d_k < a_i$ for $j = 1, \dots, l$ and $r_{j-1} < i \leq r_j$. This results in a linear combination with integer coefficients of expressions of the form

$$\frac{1}{z^{d_1+d_2+\dots+d_l}} \int_{[0,1]^m} \frac{\prod_{i=1}^m x_i^{a'_i-1} (1-x_i)^{b_i-a_i-1}}{\prod_{j=1}^l (1-zx_1x_2\dots x_{r_j})^{c'_j}} dx_1 dx_2 \dots dx_m,$$

with parameters c'_j , satisfying $0 \leq c'_j \leq c_j$, $a'_i = a_i - \sum_{k=j}^l d_k \geq 1$ for $j = 1, \dots, l$ and $r_{j-1} < i \leq r_j$. Application of Lemma 4 (in the lemma, Δ appears as P) to each such integral completes the proof.

Corollary 1 *Let the integral $S(z)$ has parameters*

$$a_i = n + 1, \quad b_i = 2n + 2, \quad c_j = n + 1.$$

Then the polynomial $D_n^{m-w(\vec{s})} P_{\vec{s}}(z)$ has integer coefficients.

Proof. Take $d_j = 0$ for $j = 1, \dots, l-1$ and $d_l = n$ in Theorem 1. Then $\Delta = n$ and all conditions of the theorem are satisfied.

We apply Corollary 1 to the integral

$$I_{2l+1,n}(z) = \int_{[0,1]^{2l+1}} \frac{\prod_{i=1}^{2l+1} x_i^n (1-x_i)^n dx_1 dx_2 \dots dx_{2l+1}}{\prod_{j=1}^l (1-zx_1 \dots x_{2j})^{n+1} (1-zx_1 x_2 \dots x_{2l} x_{2l+1})^{n+1}}.$$

By [1, Theorem 6],

$$I_{2l+1,n}(z) = \sum_{k=0}^l P_k(z^{-1}) \text{Le}_{\{2\}_k,1}(z) + \sum_{k=0}^{l-1} T_k(z^{-1}) \text{Le}_{1,\{2\}_k,1}(z) - U(z^{-1}),$$

($\{a\}_k$ means $\{a, \dots, a\}$, k times repeated) where P_k, T_k, U are polynomials with rational coefficients and $P_0(1) = 0, T_k(1) = 0$. From Corollary 1, we conclude that these polynomials multiplied by D_n^{2l+1} have integer coefficients. Letting $z \rightarrow 1-$ and using the equalities $\text{Le}_{\{2\}_k,1}(1) = 2\zeta(2k+1)$ (see [7]) and (3), this proves Vasiliev's conjecture (2).

4 Estimate of linear form coefficients

It is important in many arithmetical applications to have upper estimates for absolute values of the linear form coefficients. In this section, we study the height of a polynomial in a linear form in generalized polylogarithms, that originates from the integral $S(z)$ (see (1)).

We start with an estimate for factorial coefficients.

Lemma 5 *For nonnegative integers a and b , the following estimate holds:*

$$\frac{1}{a+b+1} \cdot \frac{(a+b)^{a+b}}{a^a b^b} \leq \frac{(a+b)!}{a!b!} \leq \frac{(a+b)^{a+b}}{a^a b^b}$$

(if $x = 0$, we let $x^x = 1$).

Proof. If $a = 0$ or $b = 0$, then both inequalities are valid. In what follows, suppose that a and b are positive integers.

Consider the Beta-integral

$$\int_0^1 x^a (1-x)^b dx = B(a+1, b+1) = \frac{a!b!}{(a+b+1)!}.$$

The function $f(x) = x^a(1-x)^b$, on the segment $[0, 1]$, achieves its maximum at the point $x = a/(a+b)$. Hence,

$$\frac{a!b!}{(a+b+1)!} \leq f\left(\frac{a}{a+b}\right) = \frac{a^a b^b}{(a+b)^{a+b}},$$

proving the first inequality. Now we prove the second inequality by induction on the value of $a+b$. The induction base $a=b=1$ is valid. Introduce the notation

$$g(a, b) = \frac{(a+b)!}{a!b!}.$$

Assuming $b > 1$, the induction hypothesis yields

$$g(a, b-1) \leq \frac{(a+b-1)^{a+b-1}}{a^a(b-1)^{b-1}}.$$

From the definition of the function g ,

$$\frac{g(a, b)}{g(a, b-1)} = \frac{a+b}{b}.$$

The function $(1+1/m)^m$ monotonically increases with m , hence

$$\left(1 + \frac{1}{a+b-1}\right)^{a+b-1} \geq \left(1 + \frac{1}{b-1}\right)^{b-1}.$$

Write the last inequality as

$$\frac{(a+b-1)^{a+b-1}}{(b-1)^{b-1}} \leq \frac{(a+b)^{a+b-1}}{b^{b-1}}$$

Thus,

$$\begin{aligned} g(a, b) &= \frac{a+b}{b} \cdot g(a, b-1) \leq \frac{a+b}{b} \cdot \frac{(a+b-1)^{a+b-1}}{a^a(b-1)^{b-1}} \\ &\leq \frac{a+b}{b} \cdot \frac{(a+b)^{a+b-1}}{a^a b^{b-1}} = \frac{(a+b)^{a+b}}{a^a b^b}, \end{aligned}$$

which is the required assertion.

Remark. Expression

$$\frac{(a+b)^{a+b}}{a^a b^b}$$

can be written as

$$\left(\frac{(\alpha + \beta)^{\alpha + \beta}}{\alpha^\alpha \beta^\beta} \right)^n,$$

where $\alpha = a/n$, $\beta = b/n$.

By [1, Lemma 2], the integral $S(z)$ is expressed as

$$S(z) = \sum_{n_1 \geq n_2 \geq \dots \geq n_l \geq 1} R(n_1, n_2, \dots, n_l) z^{n_1 - 1},$$

where

$$\begin{aligned} R(\zeta_1, \zeta_2, \dots, \zeta_l) &= \frac{\prod_{i=1}^m \Gamma(b_i - a_i)}{\prod_{j=1}^l \Gamma(c_j)} \\ &\times \frac{\prod_{j=1}^l [(\zeta_j - \zeta_{j+1} + 1)(\zeta_j - \zeta_{j+1} + 2) \dots (\zeta_j - \zeta_{j+1} + c_j - 1)]}{\prod_{j=1}^l \prod_{i=r_{j-1}+1}^{r_j} [(\zeta_j + a_i - 1)(\zeta_j + a_i) \dots (\zeta_j + b_i - 2)]}, \end{aligned}$$

To the end of the section, suppose that the parameters a_i , b_i , c_j depend linearly on an increasing parameter n , i.e.

$$a_i = \alpha_i n + \alpha'_i, \quad b_i = \beta_i n + \beta'_i, \quad c_j = \gamma_j n + \gamma'_j, \quad \alpha_i, \beta_i, \gamma_j \in \mathbb{N}, \quad \alpha'_i, \beta'_i, \gamma'_j \in \mathbb{Z}.$$

As before, $q_j = \sum_{i=r_{j-1}+1}^{r_j} (b_i - a_i)$. We also use notation

$$\begin{aligned} p_j &= \min_{r_{j-1}+1 \leq i \leq r_j} a_i - 1, & P_j &= \max_{r_{j-1}+1 \leq i \leq r_j} b_i - 2, \\ h_j &= \min_{r_{j-1}+1 \leq i \leq r_j} \alpha_i, & H_j &= \max_{r_{j-1}+1 \leq i \leq r_j} \beta_i, \\ \varphi(x, y) &= |x + y|^{x+y} \cdot |x|^{-x}. \end{aligned}$$

Here and in what follows, $|x|^x = 1$ if $x = 0$, that agrees with the limit value of $|x|^x$ as $x \rightarrow 0$.

Lemma 6 *Let $c_1 \leq q_1$ and $c_{j-1} + c_j \leq q_j$ for $j = 2, \dots, l$. Then*

$$R(\zeta_1, \zeta_2, \dots, \zeta_l) = \sum_{\vec{s}, \vec{k}} A_{\vec{s}, \vec{k}} \prod_{j=1}^l \frac{1}{(\zeta_j + k_j)^{s_j}},$$

and

$$|A_{\vec{s}, \vec{k}}| \leq (F(x_1, \dots, x_l))^{n+o(n)}, \quad n \rightarrow \infty,$$

where

$$x_j = \frac{k_j - p_j}{P_j - p_j} \in [0, 1]$$

and

$$\begin{aligned} F(x_1, \dots, x_l) &= \prod_{j=1}^l \prod_{i=r_{j-1}+1}^{r_j} \frac{(\beta_i - \alpha_i)^{\beta_i - \alpha_i}}{\varphi(\alpha_i - h_j - (H_j - h_j)x_j, \beta_i - \alpha_i)} \\ &\times \prod_{j=1}^{l-1} \frac{\varphi(h_{j+1} + (H_{j+1} - h_{j+1})x_{j+1} - h_j - (H_j - h_j)x_j, \gamma_j)}{\gamma_j^{\gamma_j}} \\ &\times \frac{\varphi(h_l + (H_l - h_l)x_l - \gamma_l, \gamma_l)}{\gamma_l^{\gamma_l}}. \end{aligned} \quad (10)$$

Proof. Expand the numerator of the function R into the sum of monomials. Consider any monomial and the corresponding function $\widehat{R}(\zeta_1, \zeta_2, \dots, \zeta_l)$. The degree of the numerator is less than the degree of the denominator in each variable in \widehat{R} . Hence, the function $R(\zeta_1, \zeta_2, \dots, \zeta_l)$ can be represented as

$$R(\zeta_1, \zeta_2, \dots, \zeta_l) = \sum_{\vec{s}, \vec{k}} A_{\vec{s}, \vec{k}} \prod_{j=1}^l \frac{1}{(\zeta_j + k_j)^{s_j}},$$

Let m_j be the maximal order of the pole in variable ζ_j . Cauchy's integral formula for a polycylindrical domain (see [9, (1.28)]), applied to partial derivatives of the function

$$(\zeta_1 + k_1)^{m_1} \cdots (\zeta_l + k_l)^{m_l} R(\zeta_1, \zeta_2, \dots, \zeta_l),$$

implies

$$\begin{aligned} A_{\vec{s}, \vec{k}} &= \frac{1}{(2\pi i)^l} \int_{|\zeta_1 + k_1| = \frac{1}{2}} \cdots \int_{|\zeta_l + k_l| = \frac{1}{2}} R(\zeta_1, \zeta_2, \dots, \zeta_l) \\ &\times (\zeta_1 + k_1)^{s_1 - 1} \cdots (\zeta_l + k_l)^{s_l - 1} d\zeta_1 \cdots d\zeta_l. \end{aligned}$$

Introduce the function

$$\Phi(u, v) = |u + v|^{\text{sign}(u+v)} \cdot |u|^{-\text{sign}(u)},$$

defined for integers u and v . The following inequality holds on the circle $|\zeta_j + k_j| = 1/2$:

$$|(\zeta_j + a_i - 1) \cdots (\zeta_j + b_i - 2)| \geq \Phi((a_i - 1) - k_j, b_i - a_i - 1)e^{o(n)},$$

$$|\zeta_l(\zeta_l + 1) \cdots (\zeta_l + c_l - 2)| \leq \Phi(k_l - c_l + 1, c_l - 1)e^{o(n)},$$

$$|(\zeta_j - \zeta_{j+1} + 1) \cdots (\zeta_j - \zeta_{j+1} + c_j - 1)| \leq \Phi(k_{j+1} - k_j, c_j - 1)e^{o(n)}.$$

We prove only the first inequality (the latter ones are proved similarly). First, consider the case k_j lying in the interval $(a_i - 1, b_i - 2)$. For $N < k_j$, we have

$$|\zeta_j + N| = |(k_j - N) - (\zeta_j + k_j)| \geq (k_j - N) - \frac{1}{2},$$

and, for $N > k_j$,

$$|\zeta_j + N| = |(N - k_j) + (\zeta_j + k_j)| \geq (N - k_j) - \frac{1}{2}.$$

Consequently (Set a product to be 1 if upper limit greater than lower),

$$\begin{aligned} |(\zeta_j + a_i - 1) \cdots (\zeta_j + b_i - 2)| &\geq \prod_{N=a_i-1}^{k_j-1} \left(k_j - N - \frac{1}{2}\right) \cdot \frac{1}{2} \cdot \prod_{N=k_j+1}^{b_i-2} \left(N - k_j - \frac{1}{2}\right) \\ &\geq \frac{1}{8} \prod_{N=a_i-1}^{k_j-2} (k_j - N - 1) \cdot \prod_{N=k_j+2}^{b_i-2} (N - k_j - 1) \\ &= \frac{(k_j - (a_i - 1))! ((b_i - 2) - k_j)!}{8(k_j - (a_i - 1))((b_i - 2) - k_j)} \\ &= \Phi((a_i - 1) - k_j, b_i - a_i - 1)e^{o(n)}. \end{aligned}$$

Now consider the case $k_j < a_i - 1$:

$$\begin{aligned} |(\zeta_j + a_i - 1) \cdots (\zeta_j + b_i - 2)| &\geq \prod_{N=a_i-1}^{b_i-2} \left(N - k_j - \frac{1}{2}\right) \geq \frac{1}{2} \prod_{N=a_i}^{b_i-2} (N - k_j - 1) \\ &= \frac{a_i - 1 - k_j}{2(b_i - 2 - k_j)} \cdot \frac{((b_i - 2) - k_j)!}{((a_i - 1) - k_j)!} \\ &= \Phi((a_i - 1) - k_j, b_i - a_i - 1)e^{o(n)}. \end{aligned}$$

The case $k_j > b_i - 2$ is considered similarly. Cases $k_j = a_i - 1$ or $k_j = b_i - 2$ can be verified by direct substitution.

Thus,

$$|A_{\vec{s}, \vec{k}}| \leq \prod_{j=1}^l \prod_{i=r_{j-1}+1}^{r_j} \frac{(b_i - a_i - 1)!}{\Phi((a_i - 1) - k_j, b_i - a_i - 1)} \\ \times \prod_{j=1}^{l-1} \frac{\Phi(k_{j+1} - k_j, c_j - 1)}{(c_j - 1)!} \cdot \frac{\Phi(k_l - c_l + 1, c_l - 1)}{(c_l - 1)!} \cdot e^{o(n)}. \quad (11)$$

Make the change $k_j = p_j + (P_j - p_j)x_j$, $j = 1, \dots, l$, $x_j \in [0, 1]$. From Lemma 5 and estimate (11) it follows that

$$|A_{\vec{s}, \vec{k}}| \leq (F(x_1, \dots, x_l))^n e^{o(n)},$$

where $F(x_1, \dots, x_l)$ is the function, specified in the lemma statement.

Theorem 2 *Let $c_1 \leq q_1$ and $c_{j-1} + c_j \leq q_j$ for $j = 2, \dots, l$. Then the heights of all polynomials in the linear form $S(z) = \sum_{\vec{s}} P_{\vec{s}}(z^{-1}) \text{Le}_{\vec{s}}(z)$ do not exceed $M^{n+o(n)}$, as $n \rightarrow \infty$, where M is the maximum of function (10) on the cube $[0, 1]^l$.*

Proof. By Lemma 6, we have the equality:

$$S(z) = \sum_{n_1 \geq n_2 \geq \dots \geq n_l \geq 1} R(n_1, n_2, \dots, n_l) z^{n_1-1} \\ = \sum_{\vec{s}, \vec{k}} A_{\vec{s}, \vec{k}} \sum_{n_1 \geq n_2 \geq \dots \geq n_l \geq 1} z^{n_1-1} \prod_{j=1}^l \frac{1}{(\zeta_j + k_j)^{s_j}}.$$

Since $s_j \leq m$ and $b_i - a_i \leq Cn$, then the number of summands in the external sum does not exceed $(m \cdot Cn)^l = e^{o(n)}$. Furthermore, consider the decomposition of the elementary sum

$$\sum_{n_1 \geq n_2 \geq \dots \geq n_l \geq 1} z^{n_1-1} \prod_{j=1}^l \frac{1}{(n_j + k_j)^{s_j}}$$

into a linear form in polylogarithms for a fixed \vec{s} and \vec{k} . The heights of all polynomials in it is $e^{o(n)}$ by Lemma 1. From Lemma 6 it follows that $|A_{\vec{s}, \vec{k}}| \leq M^{n+o(n)}$. This implies the statement of theorem.

References

- [1] ZLOBIN S.A., *Expanding multiple integrals into linear forms* // Matem. Zametki. 2005. V. 77. No. 5. P. 683–706.
- [2] VASILYEV D.V., *On small linear forms for the values of the Riemann zeta-function at odd integers* // Preprint No. 1 (558). Minsk: Nat. Acad. Sci. Belarus, Institute Math., 2001.
- [3] BEUKERS F., *A note on the irrationality of $\zeta(2)$ and $\zeta(3)$* // Bull. London Math Society. 1979. V. 11. No. 3. P. 268–272.
- [4] ZUDILIN W.V., *Well-poised hypergeometric series and multiple integrals* // Uspehi Matem. Nauk. 2002. V. 57. No. 4. P. 177–178.
- [5] KRATTENTHALER C., RIVOAL T., *Hypergéométrie et fonction zêta de Riemann* // Preprint (December 2004), submitted for publication; // <http://arxiv.org/abs/math/0311114>.
- [6] ZLOBIN S.A., *Integrals expressible as linear forms in generalized polylogarithms* // Matem. Zametki. 2002. V. 71. No. 5. P. 782–787.
- [7] ZLOBIN S.A., *Generating functions for multiple zeta values* // Vestnik MGU. Ser. 1. Matem., Mekh. 2005. No. 2. P. 55–59.
- [8] PRASOLOV V.V., *Polynomials* // M.: MCCME, 1999.
- [9] FUCHS B.A., *Introduction to the theory of analytic functions of several complex variables* // M.: Gos. Izd. Fiz.-mat. Lit., 1962.