# CERTAIN FORMULAE FOR VALUES OF THE RIEMANN ZETA FUNCTION AT INTEGRAL POINTS[1]

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Set $\zeta(s)=\sum_{m=1}^{\infty} m^{-s}$, $I_{0,n}(\sigma)=1$, and

$$I_{k,n}(\sigma)=\int_0^1\cdots\int_0^1 \frac{[x_1\dots x_k(1-x_1)\dots(1-x_k)]^n[x_1(1-x_2(\dots(1-x_k)\dots)]^\sigma\,\mathrm{d}\overline{x}}{(1-x_1(1-x_2(1-\dots(1-x_{k-1}(1-x_k)\dots)^n}.$$

It can be easily shown that $J_{2,n}(0)$ and $J_{3,n}(0)$, $n=0,1,2,\dots$, coincide with the sequences from [1] that were used there to prove that $\zeta(2)$ and $\zeta(3)$ are irrational (see also [2]).

**Theorem.**

1) $$I_{2M,0}(\sigma)=\sum_{k=1}^{M}(-1)^{k-1}\sum_{j_1,\dots,j_k=1}^{\infty}\frac{1}{(\sigma+j_1)^2\dots(\sigma+j_1+\dots+j_k)^2}I_{2M-2k,0}(0);$$

2) $$I_{2M+1,0}(\sigma)=\sum_{k=1}^{M}(-1)^{k-1}\sum_{j_1,\dots,j_k=1}^{\infty}\frac{1}{(\sigma+j_1)^2\dots(\sigma+j_1+\dots+j_k)^2}I_{2M+1-2k,0}(0)$$
$$+(-1)^{M-1}\sum_{j_1,\dots,j_{M-1}=1}^{\infty}\frac{1}{(\sigma+j_1)^2\dots(\sigma+j_1+\dots+j_{M-1})^2}\int_0^1\frac{1-x^{\sigma+j_1+\dots+j_{M-1}}}{1-x}\,\mathrm{d}x;$$

3) $I_{2,0}(0)=\zeta(2)$, $\quad I_{3,0}(0)=2\zeta(3)$, $\quad I_{4,0}(0)=\dfrac{\zeta(2)^2+\zeta(4)}{2}$, $\quad I_{5,0}(0)=2\zeta(5)$;

4) $I_{2M,0}(0)=T_M\big(\zeta(2),\zeta(4),\dots,\zeta(2M)\big)$, $\qquad T_M(x_1,\dots,x_M)\in\mathbb{Q}[x_1,\dots,x_M]$.

## References


1. F. Beukers, *A note on the irrationality of* $\zeta(2)$ *and* $\zeta(3)$, Bull. London Math. Soc. **11** (1979), no. 3, 268–272.
2. O. N. Vasilenko, *On methods for proving the irrationality of* $\zeta(3)$, Transcendence number theory and its applications, Reports of the Soviet conference (Moscow, February 2–4, 1983), MSU Publications, Moscow, 1983, pp. 15–18. (Russian)


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[1] *Number theory and its applications*, Proceedings of the science-theoretical conference (Tashkent, September 26–28, 1990), Tashkent, 1990, p. 27. (Russian)