CLAUSEN-TYPE IDENTITIES ARISING FROM THE THEORY OF MODULAR FORMS

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The binomial theorem implies the algebraicity of

$$f_1(x) = \sum_{n=0}^{\infty} \binom{2n}{n} x^n.$$

For $k \geq 2$ the series

$$f_k(x) = \sum_{n=0}^{\infty} \binom{2n}{n}^k x^n$$

are transcendental. Surprisingly enough, we have an algebraic relation between $f_2(x)$ and $f_3(x)$:

$$\left(\sum_{n=0}^{\infty} \binom{2n}{n}^2 x^n \right)^2 = \sum_{n=0}^{\infty} \binom{2n}{n}^3 \left(x(1-16x) \right)^n,$$

and no further algebraic relations are known for $f_k(x)$.

The last identity (perhaps already known to Euler) is a special case of a general identity discovered by T. Clausen: the square of a certain hypergeometric ${}_2F_1$ function can be expressed in terms of a hypergeometric ${}_3F_2$ function.

To make the things specific, define the (generalized) hypergeometric function:

$$
{}_mF_{m-1}\left(\begin{matrix} a_1,\, a_2,\, \dots,\, a_m \\ b_2,\, \dots,\, b_m \end{matrix} \,\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_m)_n}{(b_2)_n \cdots (b_m)_n} \frac{z^n}{n!},
$$

where

$$
(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1)\cdots(a+n-1) & \text{if } n \ge 1, \\ 1 & \text{if } n = 0. \end{cases}
$$

The natural domain of convergence is $|z| < 1$.

It satisfies the hypergeometric differential equation:

$$
\left(\theta \prod_{j=2}^{m} (\theta + b_j - 1) - z \prod_{j=1}^{m} (\theta + a_j)\right) y = 0, \qquad \theta = z\frac{\mathrm{d}}{\mathrm{d}z}.
$$

Clausen's identity:

$$
{}_2F_1\left(\begin{matrix} 2a,\, 2b \\ a+b+\frac{1}{2} \end{matrix} \,\middle|\, z\right)^2 = {}_3F_2\left(\begin{matrix} 2a,\, 2b,\, a+b \\ a+b+\frac{1}{2},\, 2a+2b \end{matrix} \,\middle|\, 4z(1-z)\right).
$$

Clausen's identity:

$$ {}_2F_1\left(\begin{matrix} 2a,\, 2b \\ a+b+\frac{1}{2} \end{matrix} \,\middle|\, z\right)^2 = {}_3F_2\left(\begin{matrix} 2a,\, 2b,\, a+b \\ a+b+\frac{1}{2},\, 2a+2b \end{matrix} \,\middle|\, 4z(1-z)\right). $$

Note that

$$ \frac{(\frac{1}{2})_n}{n!} = 2^{-2n}\binom{2n}{n}, $$

so that the particular case $a = b = \frac{1}{4}$ of Clausen's identity leads to the identity

$$ \left(\sum_{n=0}^{\infty}\binom{2n}{n}^2 x^n\right)^2 = \sum_{n=0}^{\infty}\binom{2n}{n}^3 \big(x(1-16x)\big)^n. $$

There are many applications of Clausen's identity. For example, by S. Ramanujan used it in his derivation of the series for $1/\pi$ like

$$ \frac{4}{\pi} = \sum_{n=0}^{\infty}\binom{2n}{n}^3 \frac{6n+1}{2^{8n}}. $$

Our interest is in the following form of Clausen's identity ($a = \frac{1}{2}c$, $b = \frac{1}{2}(1-c)$):

$$\left(\sum_{n=0}^{\infty} \frac{(c)_n(1-c)_n}{n!^2} z^n\right)^2 = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{(c)_n(1-c)_n}{n!^2} \big(z(1-x)\big)^n.$$

Are there more examples of sequences $\{u_n\} = \{u_n\}_{n=0}^{\infty}$ which are not covered by the identity but do satisfy (Clausen-type) identities

$$\left(\sum_{n=0}^{\infty} u_n z^n\right)^2 = s(z) \sum_{n=0}^{\infty} \binom{2n}{n} u_n r(z)^n,$$

where $s(z)$ and $r(z)$ are rational (or algebraic) functions of z?

Do they correspond to (Ramanujan-type) formulae for $1/\pi$?

The both questions are answered in affirmative. We have found three sequences:

$$a_n=\sum_{k=0}^{n}\binom{n}{k}^3, \quad b_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k},$$

$$\text{and} \quad c_n=\sum_{k=0}^{n}\binom{n}{k}(-8)^{n-k}\sum_{j=0}^{k}\binom{k}{j}^3.$$

Then

$$\left(\sum_{n=0}^{\infty}a_nz^n\right)^2=\frac{1}{1+8z^2}\sum_{n=0}^{\infty}\widetilde{a}_n\left(\frac{z(1+z)(1-8z)}{(1+8z^2)^2}\right)^n,$$

$$\left(\sum_{n=0}^{\infty}b_nz^n\right)^2=\frac{1}{1-9z^2}\sum_{n=0}^{\infty}\widetilde{b}_n\left(\frac{z(1-9z)(1-z)}{(1-9z^2)^2}\right)^n, \quad \text{and}$$

$$\left(\sum_{n=0}^{\infty}c_nz^n\right)^2=\frac{1}{1-72z^2}\sum_{n=0}^{\infty}\widetilde{c}_n\left(\frac{z(1+8z)(1+9z)}{(1-72z^2)^2}\right)^n$$

in the notation $\widetilde{u}_n=\binom{2n}{n}u_n$.

And we have found a plenty of Ramanujan-type formulae involving the corresponding sequences $\widetilde{u}_n = \binom{2n}{n} u_n$; the particular entries are

$$\frac{3\sqrt{2}}{\pi} = \sum_{n=0}^{\infty} \frac{\widetilde{a}_n(5n+1)}{96^n}, \qquad \frac{50\sqrt{39}}{\pi} = \sum_{n=0}^{\infty} \frac{\widetilde{a}_n(918n+99)}{10400^n},$$

$$\frac{25}{\sqrt{3}\pi} = \sum_{n=0}^{\infty} \frac{\widetilde{b}_n(16n+3)}{100^n}, \qquad \frac{75}{16\sqrt{2}\pi} = \sum_{n=0}^{\infty} \frac{\widetilde{b}_n(7n+1)}{900^n},$$

$$\frac{\sqrt{6}}{\pi} = \sum_{n=0}^{\infty} \frac{\widetilde{c}_n(5n+1)}{288^n}, \qquad \frac{32\sqrt{51}}{\pi} = \sum_{n=0}^{\infty} \frac{\widetilde{c}_n(770n+73)}{39168^n},$$

where

$$\widetilde{a}_n = \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^3, \quad \widetilde{b}_n = \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k},$$

$$\text{and} \quad \widetilde{c}_n = \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k} (-8)^{n-k} \sum_{j=0}^{k} \binom{k}{j}^3.$$

The main reason behind all these mysteriously beautiful identities is the "modular origin" of the generating series

$$f(z) = \sum_{n=0}^{\infty} u_n z^n$$

and their companions

$$\widetilde{f}(z) = \sum_{n=0}^{\infty} \binom{2n}{n} u_n z^n$$

in the following sense: in each case there are two modular functions $z(\tau)$ and $\widetilde{z}(\tau)$ (with respect to certain arithmetic subgroups of $SL_2(\mathbb{Z})$) such that $f\big(z(\tau)\big)$ and $\widetilde{f}\big(\widetilde{z}(\tau)\big)$ are modular forms of weight 1 and 2, respectively.

Skipping details of the proofs, the following modular parametrisations are used in proving the identity

$$\left(\sum_{n=0}^{\infty} b_n z^n\right)^2 = \frac{1}{1-9z^2} \sum_{n=0}^{\infty} \widetilde{b}_n \left(\frac{z(1-9z)(1-z)}{(1-9z^2)^2}\right)^n,$$

$$\text{where} \quad b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \quad \text{and} \quad \widetilde{b}_n = \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}.$$

We take

$$z(\tau) = \frac{\eta^4(\tau)\eta^8(6\tau)}{\eta^4(3\tau)\eta^8(2\tau)}, \qquad f(\tau) = \frac{\eta^6(2\tau)\eta(3\tau)}{\eta^2(6\tau)\eta^3(\tau)},$$

$$\widetilde{z}(\tau) = \frac{\eta^4(\tau)\eta^4(2\tau)\eta^4(3\tau)\eta^4(6\tau)}{(\eta^4(\tau)\eta^4(2\tau) + 9\eta^4(3\tau)\eta^4(6\tau))^2},$$

$$\widetilde{f}(\tau) = \frac{1}{4}\big(6P(6\tau) + 2P(2\tau) - P(\tau) - 3P(3\tau)\big)$$

in the standard notation

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1 - q^n) \quad \text{and} \quad P(\tau) = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1 - q^n},$$

with $q = e^{2\pi i\tau}$.

Then

$$f(\tau) = \sum_{n=0}^{\infty} b_n z(\tau)^n \quad \text{and} \quad \widetilde{f}(\tau) = \sum_{n=0}^{\infty} \widetilde{b}_n \widetilde{z}(\tau)^n.$$

The modular origin gives rise to some further remarkable properties of the sequences. For example, we have the following arithmetic congruences:

$$a_{np} \equiv a_n \pmod{p^3}, \quad b_{np} \equiv b_n \pmod{p^2}, \quad \text{and} \quad c_{np} \equiv c_n \pmod{p^2},$$

where p is an arbitrary prime, and the same ones for $\widetilde{a}_n$, $\widetilde{b}_n$, and $\widetilde{c}_n$. They remind about the congruence

$$u_{np} \equiv u_n \pmod{p^3}$$

(proved by I. Gessel in 1982) valid for the sequence of Apéry's numbers

$$u_n = \sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2$$

used by R. Apéry in his famous proof of the irrationality of $\zeta(3)$.

Another relation with the sequence of Apéry's numbers is the fact that the so-called Legendre transform of the sequences

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^3 \quad \text{and} \quad b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}$$

is given by Apéry's sequence:

$$\sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n+k}{k} \binom{n}{k} a_k = \sum_{k=0}^{n} \binom{n+k}{k} \binom{n}{k} (-1)^{n-k} b_k.$$

The generating series for the sequences

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^3, \quad b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k},$$

$$\text{and} \quad c_n = \sum_{k=0}^{n} \binom{n}{k} (-8)^{n-k} \sum_{j=0}^{k} \binom{k}{j}^3$$

satisfy the differential equations

$$\big(\theta^2 - z(A\theta^2 + A\theta + B) + Cz^2(\theta+1)^2\big)y = 0, \quad \text{where } \theta = z\frac{\mathrm{d}}{\mathrm{d}z},$$

with $(A, B, C) = (7, 2, -8)$, $(10, 3, 9)$, and $(-17, -6, 72)$, respectively. (This differential equation is more general than the 2nd order hypergeometric equation.)

This is equivalent to saying that the sequences themselves satisfy the Apéry-like recurrence

$$(n+1)^2 u_{n+1} - (An^2 + An + B)u_n + Cn^2 u_{n-1} = 0 \quad \text{for } n = 0, 1, 2, \dots .$$

It has been pointed to us by G. Almkvist that having the above differential equation

$$\big(\theta^2 - z(A\theta^2 + A\theta + B) + Cz^2(\theta+1)^2\big)y = 0, \quad \text{where } \theta = z\frac{\mathrm{d}}{\mathrm{d}z},$$

for $f(z) = \sum_{n=0}^{\infty} u_n z^n$ in mind it is possible to derive the Clausen-type identity using a routine computation in `Maple`. The corresponding series $\widetilde{f}(z) = \sum_{n=0}^{\infty} \widetilde{u}_n z^n$ satisfies the 3rd order equation

$$\big(\theta^3 - 2z(2\theta+1)(A\theta^2 + A\theta + b) + 4Cz^2(\theta+1)(2\theta+1)(2\theta+3)\big)y = 0,$$

and it is a `Maple` exercise to check that

$$f(z)^2 = \frac{1}{1 - Cz^2} \cdot \widetilde{f}\bigg(\frac{z(1 - Az + Cz^2)}{(1 - Cz^2)^2}\bigg).$$

This is a general form of the identities given above, which is meaningless without explicit formulae for the coefficients of the series $f(z)$.

An interesting problem which we attack in a joint project with G. Almkvist and D. van Straten is finding analogues of Clausen's identity for (hypergeometric) differential equations of order 4 and 5. But this is a completely different story because no modularity is available in that cases.