Hypergeometric approximations to polylogarithms

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Abstract

We present three applications of the hypergeometric method to the arithmetic study of polylogarithm values and, in particular, of zeta values. Part 1 is joint work with Khodabakhsh and Tatiana Hessami Pilehrood on the irrationality of certain numbers involving di- and trilogarithms. In Part 2 we give three hypergeometric constructions leading to simultaneous approximations to $\zeta(2)$ and $\zeta(3)$. Part 3 contains a curious hypergeometric identity for the known rational approximations to $\zeta(4)$; the identity suggests an extension of the hypergeometric technique which might lead to a natural way to prove the so-called Denominator Conjectures for zeta values.

It seems that a dominating technique in the study of arithmetic properties of values of the polylogarithmic functions

$$\mathrm{Li}_j(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^j}, \quad |z| < 1, \qquad j = 1, 2, 3, \dots,$$

and, in particular, of zeta values $\zeta(j) = \mathrm{Li}_j(1)$ for $j \ge 2$, is the hypergeometric one. Approximations to the functions can be represented by means of classical hypergeometric series as well as their integral and multiple extensions. In this lecture I touch three applications of the hypergeometric technique. Part 1 is joint work with Khodabakhsh and Tatiana Hessami

[1] A talk at the conference "Diophantine approximation and transcendental numbers" (CIRM, Marseille Luminy, September 4–8, 2006).

Pilehrood on the irrationality of certain numbers involving values of $\mathrm{Li}_2(z)$ and $\mathrm{Li}_3(z)$. In Part 2 I give three different hypergeometric constructions leading to simultaneous approximations to $\zeta(2)$ and $\zeta(3)$. Although this does not result in the linear independence of the numbers with 1 over $\mathbb{Q}$, I obtain new rational approximations to $\zeta(2)$ and $\zeta(3)$ yielding their irrationality. Finally, Part 3 contains a new hypergeometric identity for the known rational approximations to $\zeta(4)$. The identity suggests an extension of the hypergeometric technique which might lead to a natural way to prove the so-called Denominator Conjectures for zeta values.

1 Irrationality of certain numbers that contain values of the di- and trilogarithm

The irrationality result proved jointly with Khodabakhsh and Tatiana Hessami Pilehrood in [4] involves the values of the functions

$$f_j(z) = \mathrm{Li}_j(z) - \mathrm{Li}_1(z) \cdot \frac{\log^{j-1} z}{(j-1)!}, \qquad j = 2, 3, \dots .$$

Recall that $\mathrm{Li}_1(z) = -\log(1-z)$.

Theorem 1. *For $z \in \{1/2, 2/3, 3/4, 4/5\}$, at least one of the two numbers $f_2(z)$ and $f_3(z)$ is irrational.*

Using the classical formulae (see, e.g., [6, equations (1.11) and (6.10)]) we may also express the functions $f_2(z)$ and $f_3(z)$ as follows:

$$f_2(z) = \frac{\pi^2}{6} - \mathrm{Li}_2(1-z),$$
$$f_3(z) = \zeta(3) + \frac{1}{6}\log^3 z + \frac{\pi^2}{6}\log z - \mathrm{Li}_3(1-z) - \mathrm{Li}_3(1-z^{-1}).$$

Moreover, the values of these functions at the point $z = 1/2$ are computed by means of $\log 2$, $\zeta(2) = \pi^2/6$ and $\zeta(3)$ (see [6, equations (1.16) and (6.12)]):

$$f_2(1/2) = \frac{1}{2}\zeta(2) + \frac{1}{2}\log^2 2, \qquad f_3(1/2) = \frac{7}{8}\zeta(3) - \frac{1}{2}\zeta(2)\log 2 - \frac{1}{3}\log^3 2.$$

Using these formulae we obtain the following corollary of Theorem 1.

Theorem 2. *At least one of the two numbers*

$$\pi^2 + 6\log^2 2 \qquad \text{and} \qquad \zeta(3) - \frac{2}{21}(\pi^2 + 4\log^2 2)\log 2$$

is irrational.

Relative results, 'at least one of the numbers $3\zeta(3) - c\zeta(2)$, $\zeta(2) - 2c\log 2$ ($c \in \mathbb{Q}$) is irrational' and 'at least one of the numbers $\mathrm{Li}_2(1/q)$, $\mathrm{Li}_3(1/q)$ ($q \in \mathbb{Z} \setminus \{0, 2\}$) is irrational', are proved in [2] and [3], respectively. The irrationality of $\mathrm{Li}_2(1/q)$ is known [9] for integers $q \le -5$ and $q \ge 6$.

Our proof relies on a general hypergeometric construction of two linear forms in the polylogarithms and positive powers of the logarithm, respectively. This idea was recently used in [1] for proving that at least one of the three numbers $f_2(1/2)$, $f_3(1/2)$ and $f_4(1/2)$ is irrational. We are able to improve this earlier result and present the related ones due to the powerful group-structure arithmetic method introduced in [7] and [8] in order to prove new estimates for irrationality measures of $\zeta(2)$ and $\zeta(3)$ (see also [9] and [12]).

As usual for the hypergeometric method, we start with a rational function $R(t)$:

$$\begin{aligned} R(\boldsymbol{a}, \boldsymbol{b}; t) &= \frac{\prod_{j=1}^{s} \Gamma(b_j - a_j)}{\Gamma(a_0)} \prod_{j=0}^{s} \frac{\Gamma(t + a_j)}{\Gamma(t + b_j)} \\ &= \frac{\prod_{j=1}^{s} (b_j - a_j - 1)!}{(a_0 - 1)!} \cdot \frac{(t+1)(t+2)\cdots(t+a_0-1)}{\prod_{j=1}^{s} (t+a_j)\cdots(t+b_j-1)} \\ &= \sum_{j=1}^{s} \sum_{k=a_j}^{b_j - 1} \frac{A_{jk}}{(t+k)^j}, \end{aligned} \tag{1}$$

where the set of integer parameters $\boldsymbol{a}, \boldsymbol{b}$ satisfies

$$\begin{gathered} b_0 = 1 < a_0 \le a_1 \le a_2 < \cdots \le a_s < b_s \le b_{s-1} \le \cdots \le b_1, \\ \sum_{j=0}^{s} a_j < \sum_{j=0}^{s} b_j, \qquad a_0 + a_s \le b_s. \end{gathered} \tag{2}$$

Then our main objects are as follows:

$$
\begin{aligned}
I = I(\boldsymbol{a}, \boldsymbol{b}; z) &= \sum_{t=1-a_0}^{\infty} R(t) z^{t+a_0} = \sum_{j=1}^{s} \sum_{k=a_j}^{b_j-1} A_{jk} z^{-k+a_0} \sum_{t=1-a_0}^{\infty} \frac{z^{t+k}}{(t+k)^j} \\
&= \sum_{j=1}^{s} \sum_{k=a_j}^{b_j-1} A_{jk} z^{-k+a_0} \biggl(\mathrm{Li}_j(z) - \sum_{l=1}^{k-a_0} \frac{z^l}{l^j} \biggr) \\
&= \sum_{j=1}^{s} \mathrm{Li}_j(z) \cdot \sum_{k=a_j}^{b_j-1} A_{jk} z^{-k+a_0} - \sum_{j=1}^{s} \sum_{k=a_j}^{b_j-1} A_{jk} \sum_{l=1}^{k-a_0} \frac{z^{-(k-a_0^*-l)}}{l^j} \\
&= \sum_{j=1}^{s} P_j(z^{-1}) \, \mathrm{Li}_j(z) - P_0(z^{-1}),
\end{aligned} \tag{3}
$$

and (the closed contour $\mathcal{L}$ below surrounds all poles $t = -k$ for $a_1 \le k < b_1$ of the rational function $R(t)$)

$$
\begin{aligned}
J = J(\boldsymbol{a}, \boldsymbol{b}; z) &= \frac{z^{a_0}}{2\pi i} \oint_{\mathcal{L}} R(t) z^t \, \mathrm{d}t = z^{a_0} \sum_{k} \mathrm{Res}_{t=-k} (R(t) z^t) \\
&= \sum_{j=1}^{s} \sum_{k} A_{jk} z^{-k+a_0} \cdot \mathrm{Res}_{t=-k} \frac{z^{t+k}}{(t+k)^j} = \sum_{j=1}^{s} \sum_{k} A_{jk} z^{-k+a_0^*} \cdot \frac{\log^{j-1} z}{(j-1)!} \\
&= \sum_{j=1}^{s} P_j(z^{-1}) \frac{\log^{j-1} z}{(j-1)!}.
\end{aligned} \tag{4}
$$

All this means that we arrange to construct 'simultaneous' approximations to the set of polylogarithms $\mathrm{Li}_1(z), \dots, \mathrm{Li}_s(z)$ and to the set of logarithm powers $\log z, \dots, \frac{1}{(s-1)!} \log^{s-1} z$. This is essentially the idea from [1, Theorem 3].

We are interested in the hypergeometric series I and the hypergeometric integral J in the special case

$$
a_j = \alpha_j n + 1, \quad j = 0, 1, \dots, n, \qquad \text{and} \qquad b_j = \beta_j n + 2, \quad j = 1, \dots, n, \tag{5}
$$

where $\alpha_0, \alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$ are positive integers. We stress that finding correct denominators of the coefficients of the involved polynomials $P_j(z^{-1})$ is based on the asymmetry of the rational function $R(t)$: any permutation of the parameters $a_0, a_1, \dots, a_s$ remains the shapes of I and J.

Computing analytic and arithmetic behaviour of I and J as $n \to \infty$, we apply the asymptotic results to the linear forms

$$L(\boldsymbol{a}, \boldsymbol{b}; z) = I(\boldsymbol{a}, \boldsymbol{b}; z) - \mathrm{Li}_1(z) J(\boldsymbol{a}, \boldsymbol{b}; z) = \sum_{j=2}^{s} P_j(z^{-1}) f_j(z) - P_0(z^{-1})$$

choosing $s = 3$. It happens that an optimal choice for the integer parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$ depends on $z \in \{1/2, 2/3, 3/4, 4/5\}$. For example, if $z = 1/2$ we take

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2, \beta_3) = (3, 4, 5, 6; 17, 16, 15).$$

For technical details, I refer to the paper [4].

2 Simultaneous approximations to $\zeta(2)$ and $\zeta(3)$

The three hypergeometric constructions below depend on an increasing integer parameter n.

First [10] I take the rational functions

$$R_n(t) = -\frac{n!^2 \prod_{j=1}^{n}(t-j)}{\prod_{j=0}^{n}(t+j)^3}, \qquad R_n'(t) = \frac{n!^2 \prod_{j=0}^{n}(t-j)}{\prod_{j=0}^{n}(t+j)^3},$$

and consider the corresponding hypergeometric series

$$r_n = \sum_{k=1}^{\infty} R_n(t)\big|_{t=\nu} = q_n \zeta(3) + p_n \zeta(2) - s_n,$$
$$r_n' = \sum_{k=1}^{\infty} R_n'(t)\big|_{t=\nu} = q_n' \zeta(3) + p_n' \zeta(2) - s_n',$$

where

$$q_n, q_n' \in \mathbb{Z}, \quad D_n p_n, D_n p_n' \in \mathbb{Z}, \quad D_n^3 s_n, D_n^3 s_n' \in \mathbb{Z}, \tag{6}$$

D_n stands for the least common multiple of the numbers $1, 2, \dots, n$ (and $D_0 = 1$). The standard eliminating argument leads one to the linear forms

$$q_n r_n' - q_n' r_n = (q_n p_n' - q_n' p_n)\zeta(2) - (q_n s_n' - q_n' s_n) = u_n \zeta(2) - v_n,$$
$$p_n' r_n - p_n r_n' = (q_n p_n' - q_n' p_n)\zeta(3) - (p_n' s_n - p_n s_n') = u_n \zeta(3) - w_n,$$

where, by (6),

$$D_n u_n \in \mathbb{Z}, \quad D_n^3 v_n \in \mathbb{Z}, \quad D_n^4 w_n \in \mathbb{Z}. \tag{7}$$

My second construction [14] is based on the rational function

$$\widetilde{R}_n(t) = \frac{((t-1)(t-2)\cdots(t-n))^3}{n!^2 \cdot t(t+1)\cdots(t+n)}.$$

Then hypergeometric approximations to the first three polylogarithms are given by the series

$$\begin{aligned}
\widetilde{r}_n(z) &= \sum_{\nu=1}^{\infty} z^\nu \widetilde{R}_n(t)\Big|_{t=\nu} = \widetilde{u}_n(z)\,\mathrm{Li}_1(z) - \widetilde{s}_n(z),\\
\widetilde{r}'_n(z) &= -\sum_{\nu=1}^{\infty} z^\nu \frac{\mathrm{d}\widetilde{R}_n(t)}{\mathrm{d}t}\bigg|_{t=\nu} = \widetilde{u}_n(z)\,\mathrm{Li}_2(z) - \widetilde{v}_n(z),\\
\widetilde{r}''_n(z) &= \frac{1}{2}\sum_{\nu=1}^{\infty} z^\nu \frac{\mathrm{d}^2\widetilde{R}_n(t)}{\mathrm{d}t^2}\bigg|_{t=\nu} = \widetilde{u}_n(z)\,\mathrm{Li}_3(z) - \widetilde{w}_n(z),
\end{aligned}$$

where

$$\widetilde{u}_n(z) = (-1)^n \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}^3 \left(-\frac{1}{z}\right)^k \tag{8}$$

and

$$\begin{gathered}
z_1^n \widetilde{u}_n(z) \in \mathbb{Z}, \quad (z_1 z_2)^n D_n \widetilde{s}_n(z) \in \mathbb{Z},\\
(z_1 z_2)^n D_n D_{2n} \widetilde{v}_n(z) \in \mathbb{Z}, \quad (z_1 z_2)^n D_n D_{2n}^2 \widetilde{w}_n(z) \in \mathbb{Z},
\end{gathered} \tag{9}$$

z_1 and z_2 denote the denominators of the numbers $1/z$ and $z/(1-z)$, respectively.

I am interested in the limiting case $z \to 1$ when one has

$$\widetilde{r}'_n(1) = \widetilde{u}_n \zeta(2) - \widetilde{v}_n, \quad \widetilde{r}''_n(1) = \widetilde{u}_n \zeta(3) - \widetilde{w}_n, \qquad n = 0, 1, \dots,$$

where for $\widetilde{u}_n = \widetilde{u}_n(1)$, $\widetilde{v}_n = \widetilde{v}_n(1)$, and $\widetilde{w}_n = \widetilde{w}_n(1)$ from (8), (9) one can obtain

$$\widetilde{u}_n \in \mathbb{Z}, \quad D_n D_{2n} \widetilde{v}_n \in \mathbb{Z}, \quad D_n D_{2n}^2 \widetilde{w}_n \in \mathbb{Z}. \tag{10}$$

Finally, we take the rational function

$$\widetilde{\widetilde{R}}_n(t) = \frac{(t-1)(t-2)\cdots(t-n)\cdot(2t-1)(2t-2)\cdots(2t-n)}{(t(t+1)(t+2)\cdots(t+n))^2}$$

and consider the following two series:

$$\frac{1}{2}\sum_{\nu=1}^{\infty}(-1)^{\nu-1}R_n(t)\Big|_{t=\nu/2}=\widetilde{\widetilde{u}}_n\zeta(2)-\widetilde{\widetilde{v}}_n,$$
$$-\frac{1}{2}\sum_{\nu=1}^{\infty}\frac{\mathrm{d}R_n(t)}{\mathrm{d}t}\bigg|_{t=\nu}=\widetilde{\widetilde{u}}_n\zeta(3)-\widetilde{\widetilde{w}}_n.$$

The explicit formulae for the approximants allow one to show that

$$\widetilde{\widetilde{u}}_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{n}\binom{n+2k}{n}\in\mathbb{Z}, \qquad \text{for} \quad n=0,1,2,\dots . \qquad (11)$$
$$D_{2n}^2\widetilde{\widetilde{v}}_n\in\mathbb{Z}, \quad D_n^3\widetilde{\widetilde{w}}_n\in\mathbb{Z},$$

Theorem 3. *For* $n=0,1,2,\dots$, *one has*

$$\binom{2n}{n}^{-1}u_n=\widetilde{u}_n=\widetilde{\widetilde{u}}_n, \quad \binom{2n}{n}^{-1}v_n=\widetilde{v}_n=\widetilde{\widetilde{v}}_n, \quad \binom{2n}{n}^{-1}w_n=\widetilde{w}_n=\widetilde{\widetilde{w}}_n, \qquad (12)$$

that is, the three hypergeometric constructions give the same sequence of simultaneous rational approximations to 1, $\zeta(2)$ *and* $\zeta(3)$.

From Theorem 3 and the inclusions (6), (10), (11) one may easily deduce that

$$\widetilde{u}_n\in\mathbb{Z}, \quad D_nD_{2n}\widetilde{v}_n\in\mathbb{Z}, \quad D_n^3\widetilde{w}_n\in\mathbb{Z}, \qquad \text{for} \quad n=0,1,2,\dots . \qquad (13)$$

Theorem 3 can be shown by means of certain hypergeometric identities. A simpler way (used in [10] and [14]) is based on the algorithm of creative telescoping. Indeed, the following statement is valid.

Theorem 4. *The above sequences* (12) *satisfy the Apéry-type polynomial recurrence relation*

$$\begin{aligned}
&2(946n^2-731n+153)(2n+1)(n+1)^3u_{n+1}\\
&\quad-2(104060n^6+127710n^5+12788n^4-34525n^3-8482n^2+3298n+1071)u_n\\
&\quad+2(3784n^5-1032n^4-1925n^3+853n^2+328n-184)nu_{n-1}\\
&\quad-(946n^2+1161n+368)n(n-1)^3u_{n-2}=0, \qquad n=2,3,\dots,
\end{aligned}$$

of order 3, *and the necessary initial data is as follows:*

$$\widetilde{u}_0=1, \quad \widetilde{u}_1=7, \quad \widetilde{u}_2=163,$$
$$\widetilde{v}_0=0, \quad \widetilde{v}_1=\frac{23}{2}, \quad \widetilde{v}_2=\frac{2145}{8}, \qquad \widetilde{w}_0=0, \quad \widetilde{w}_1=\frac{17}{2}, \quad \widetilde{w}_2=\frac{3135}{16}.$$

In addition,

$$\limsup_{n\to\infty}|\widetilde{r}'_n|^{1/n}=\limsup_{n\to\infty}|\widetilde{r}''_n|^{1/n}=|\lambda_{1,2}|=0.067442248\ldots,$$
$$\lim_{n\to\infty}|\widetilde{u}_n|^{1/n}=\lim_{n\to\infty}|\widetilde{v}_n|^{1/n}=\lim_{n\to\infty}|\widetilde{w}_n|^{1/n}=\lambda_3=54.96369509\ldots,$$

where $\lambda_{1,2}=0.018152450\ldots\pm i0.064953409\ldots$ *and* λ_3 *are zeros of the characteristic polynomial* $4\lambda^3-220\lambda^2+8\lambda-1$.

Since $\log|\lambda_{1,2}|=-2.69648361\ldots>-3$, from (13) and Theorem 4 we cannot conclude about the irrationality of either $\zeta(2)$ or $\zeta(3)$. However, the use of an asymmetric rational function

$$\begin{aligned}R(t)&=R(\boldsymbol{a},\boldsymbol{b};t)\\&=\frac{(2t+b_0)(2t+b_0+1)\cdots(2t+a_0-1)}{(a_0-b_0)!}\cdot\frac{(t+b_1)\cdots(t+a_1-1)}{(a_1-b_1)!}\\&\qquad\times\frac{(b_2-a_2-1)!}{(t+a_2)\cdots(t+b_2-1)}\cdot\frac{(b_3-a_3-1)!}{(t+a_3)\cdots(t+b_3-1)}\\&=\frac{(b_2-a_2-1)!\,(b_3-a_3-1)!}{(a_0-b_0)!\,(a_1-b_1)!}\cdot\frac{\Gamma(2t+a_0)\,\Gamma(t+a_1)\,\Gamma(t+a_2)\,\Gamma(t+a_3)}{\Gamma(2t+b_0)\,\Gamma(t+b_1)\,\Gamma(t+b_2)\,\Gamma(t+b_3)},\end{aligned}$$

where the integers $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfy

$$b_1=1<a_1,a_2,a_3<b_2,b_3,\quad b_0<a_0\le 2\max\{a_1,a_2,a_3\},$$
$$a_0+a_1+a_2+a_3\le b_0+b_1+b_2+b_3+2,$$

lead to the following curious application.

Taking

$$a_0=10n+\tfrac12,\quad a_1=6n+1,\quad a_2=7n+1,\quad a_3=8n+1,$$
$$b_0=6n+1,\quad b_1=1,\quad b_2=13n+2,\quad b_3=12n+2,$$

for the coefficients of linear forms

$$r_n=\sum_{\nu=-10n}^{\infty}(-1)^\nu R(t)\big|_{t=\nu/2}=u_n\zeta(2)-v_n,$$
$$r'_n=\sum_{\nu=-5n}^{\infty}\frac{\mathrm{d}R(t)}{\mathrm{d}t}\bigg|_{t=\nu}=u_n\zeta(3)-w_n,$$

we obtain the inclusions

$$\Phi_n^{-1}u_n\in\mathbb{Z},\quad D_{8n}D_{16n}\Phi_n^{-1}v_n\in\mathbb{Z},\quad D_{8n}^3\Phi_n^{-1}w_n\in\mathbb{Z},$$

where Φ_n is a certain product over primes,

$$\lim_{n\to\infty} \frac{\log \Phi_n}{n} = 8.48973583\dots.$$

On the other hand,

$$\limsup_{n\to\infty} \frac{\log |r_n|}{n} = \limsup_{n\to\infty} \frac{\log |r'_n|}{n} = -17.610428885\dots.$$

Thus, the linear forms r_n and r'_n allow one to deduce the irrationality of $\zeta(2)$ and $\zeta(3)$, respectively, but not on their $\mathbb{Q}$-linear independence with 1 (the common denominator of the coefficients is $D_{8n}^2 D_{16n} \Phi_n^{-1}$).

3 A hypergeometric identity related to rational approximations to $\zeta(4)$

For each $n = 0, 1, 2, \dots$, consider the following two rational functions:

$$R_n(t) = (-1)^n \Bigl(t + \frac{n}{2}\Bigr) \frac{\prod_{l=1}^n (t-l)^2 \cdot \prod_{l=1}^n (t+n+l)^2}{\prod_{l=0}^n (t+l)^4} \tag{14}$$

and

$$\widetilde{R}_n(t) = \frac{n! \prod_{l=1}^n (t-l)}{\prod_{l=0}^n (t+l)^2} \sum_{j=0}^n \binom{n}{j}^2 \binom{n+j}{n} \frac{\prod_{l=0}^{n-1} (t-j+l)}{n!}. \tag{15}$$

Problem 1. Prove that the following equality is valid for any $n \ge 0$:

$$-\frac{1}{3} \sum_{\nu=1}^{\infty} \frac{\mathrm{d}R_n(t)}{\mathrm{d}t}\bigg|_{t=\nu} = \frac{1}{3} \sum_{\nu=1}^{\infty} \frac{\mathrm{d}^2 \widetilde{R}_n(t)}{\mathrm{d}t^2}\bigg|_{t=\nu}. \tag{16}$$

The series on the left-hand side is the familiar sequence $u_n\zeta(4) - v_n$, $n = 0, 1, 2, \dots$, of rational approximations to $\zeta(4)$; both the u_n and v_n satisfy the Apéry-like recursion [11]

$$\begin{aligned}(n+1)^5 u_{n+1} - 3(2n+1)(3n^2+3n+1)(15n^2+15n+4)u_n \\ - 3n^3(3n-1)(3n+1)u_{n-1} = 0 \qquad \text{for} \quad n \ge 1,\end{aligned} \tag{17}$$

with the initial data $u_0 = 1$, $u_1 = 12$ and $v_0 = 0$, $v_1 = 13$.

Concerning the right-hand side of (16), write

$$\widetilde{R}_n(t) = \sum_{k=0}^n \biggl(\frac{A_k^{(n)}}{(t+k)^2} + \frac{B_k^{(n)}}{t+k} \biggr) \tag{18}$$

and use the standard routine to show that

$$\frac{1}{3}\sum_{\nu=1}^{\infty}\frac{\mathrm{d}^2\widetilde{R}_n(t)}{\mathrm{d}t^2}\bigg|_{t=\nu}=\widetilde{u}_n\zeta(4)-\widetilde{v}_n, \tag{19}$$

where

$$\widetilde{u}_n=\sum_{k=0}^{n}A_k^{(n)}=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{n}\sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{n}\binom{k+j}{n}, \tag{20}$$

$$\widetilde{v}_n=\sum_{k=0}^{n}\sum_{l=1}^{k}\biggl(\frac{A_k^{(n)}}{l^4}+\frac{B_k^{(n)}}{3l^3}\biggr). \tag{21}$$

The equality $u_n=\widetilde{u}_n$ (cf. (20)) for any $n\ge 0$ was first established in [5] (see also [13]). I have verified the equality $v_n=\widetilde{v}_n$ (using the recursion for v_n and the representation (21) for $\widetilde{v}_n$) up to $n=20$. This has led me to the expectation (16). It can be shown by the algorithm of creative telescoping for double hypergeometric series. However one can hardly learn from this algorithmic proof how to generalize identity (16).

The advantage of the representation on the left-hand side of (16) is a simplicity of computing analytic aspects of the approximations $u_n\zeta(4)-v_n$ as $n\to\infty$. From (18)–(21) one easily obtains the arithmetic information

$$\Phi_n^{-1}\cdot\widetilde{u}_n\in\mathbb{Z},\quad \Phi_n^{-1}\cdot 3D_n^4\widetilde{v}_n\in\mathbb{Z},\qquad n=0,1,2,\dots, \tag{22}$$

where D_n denotes the least common multiple of the numbers $1,2,\dots,n$ and $\Phi_n=\prod_p p^{\lfloor\nu_p/2\rfloor}$ with $\nu_p=\nu_p^{(n)}=\operatorname{ord}_p(3n)!/n!^3$. The known way [5] of deducing inclusions (22) for the left-hand side of (16) is very-very sophisticated...

A puzzling thing is that the series on the right-hand side of (16) does not look tending to 0 as $n\to\infty$. In fact, analytic investigation of the series and its leading coefficient $\widetilde{u}_n$ seems to be beyond the reach. The moral is: use the left-hand side of (16) for establishing the analytic behaviour and the right-hand side for the arithmetic one.

Of course, the main goal of showing (16) is the following

Problem 2. Find and prove an appropriate analogue of identity (16) for the general linear approximations to $\zeta(4)$ considered in [11]. Use it to deduce the general denominator conjecture from [11].

I would expect that there are plenty of other identities of similar kind, especially for the well-poised linear forms in zeta values treated in [5]. This seems to be a promising way of proving general denominator conjectures.

Until the last summer I was pretty sure that all reasonable approximations to the values of polylogarithms are produced by summing a rational function glued from the elementary rational bricks

$$\frac{\prod_{j=a}^{b-1}(t+j)}{(b-a)!} \qquad \text{and} \qquad \frac{(b-a-1)!}{\prod_{j=a}^{b-1}(t+j)}.$$

Looking on the right-hand side of (16) I see how mistaken I was.

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